The quantum H_3 and H_4 integrable systems

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The H_3 rational model

The H_3 rational Hamiltonian is

$$\mathcal{H}_{H_3} = \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{g}{x_k^2}\right] + \sum_{\{i,j,k\}}\sum_{\mu_{1,2}=0,1}\frac{2g}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]^2}$$

where $\{i,j,k\} = \{1,2,3\}$ and all even permutations. The coupling constant is

$$g = \nu(\nu-1) > -\frac{1}{4}$$

and

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Explicitly:

$$\mathcal{H}_{H_3} = -\frac{1}{2}\Delta^{(3)} + \frac{1}{2}\omega^2(x_1^2+x_2^2+x_3^2) + \frac{1}{2}\nu(\nu-1)\left[\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}\right]$$
$$+2\nu(\nu-1)\left[\frac{1}{(x_1+\varphi_+x_2+\varphi_-x_3)^2}+\frac{1}{(x_1-\varphi_+x_2+\varphi_-x_3)^2}\right.$$
$$+\frac{1}{(x_1+\varphi_+x_2-\varphi_-x_3)^2}+\frac{1}{(x_1-\varphi_+x_2-\varphi_-x_3)^2}+\frac{1}{(x_2+\varphi_+x_3+\varphi_-x_1)^2}$$
$$+\frac{1}{(x_2-\varphi_+x_3+\varphi_-x_1)^2}+\frac{1}{(x_2+\varphi_+x_3-\varphi_-x_1)^2}+\frac{1}{(x_2-\varphi_+x_3-\varphi_-x_1)^2}$$
$$+\frac{1}{(x_3+\varphi_+x_1+\varphi_-x_2)^2}+\frac{1}{(x_3-\varphi_+x_1+\varphi_-x_2)^2}+\frac{1}{(x_3+\varphi_+x_1-\varphi_-x_2)^2}$$
$$\left.+\frac{1}{(x_3-\varphi_+x_1-\varphi_-x_2)^2}\right]$$

The Hamiltonian is invariant wrt the H_3 Coxeter group, which is the full symmetry group of the icosahedron. It is a subgroup of $O(3)$ and has order 120.

The Hamiltonian is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j$$
$$\varphi_+ \longleftrightarrow \varphi_-$$

The ground state function and its eigenvalue are

$$\Psi_0 = \Delta_1^{\nu}\Delta_2^{\nu}\exp\left(-\frac{\omega}{2}\sum_{k=1}^{3}x_k^2\right) \ , \quad E_0 = \frac{3}{2}\omega(1+10\nu)$$

where

$$\Delta_1 = \prod_{k=1}^{3} x_k$$

$$\Delta_2 = \prod_{\{i,j,k\}}\ \prod_{\mu_{1,2}=0,1} [x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]$$

Explicitly:

$$\Psi_0 = [x_1\ x_2\ x_3]^\nu \times$$
$$[(x_1 + \varphi_+ x_2 + \varphi_- x_3)(x_1 - \varphi_+ x_2 + \varphi_- x_3)(x_1 + \varphi_+ x_2 - \varphi_- x_3)$$
$$(x_1 - \varphi_+ x_2 - \varphi_- x_3)(x_2 + \varphi_+ x_3 + \varphi_- x_1)(x_2 - \varphi_+ x_3 + \varphi_- x_1)$$
$$(x_2 + \varphi_+ x_3 - \varphi_- x_1)(x_2 - \varphi_+ x_3 - \varphi_- x_1)(x_3 + \varphi_+ x_1 + \varphi_- x_2)$$
$$(x_3 - \varphi_+ x_1 + \varphi_- x_2)(x_3 + \varphi_+ x_1 - \varphi_- x_2)(x_3 - \varphi_+ x_1 - \varphi_- x_2)]^\nu$$
$$\times \exp\left[-\frac{\omega}{2}(x_1^2 + x_2^2 + x_3^2)\right]$$

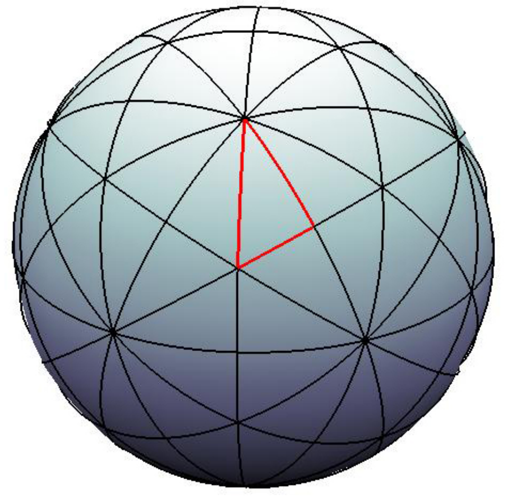
Configuration space

The configuration space is the domain in $\mathbf{R}^3$ where $x_{1,2,3} > 0$ bounded by the planes

$$x_1 = 0 \ , \quad x_3 = 0 \ ,$$

$$x_3 + \varphi_+ x_1 + \varphi_- x_2 = 0 \ .$$

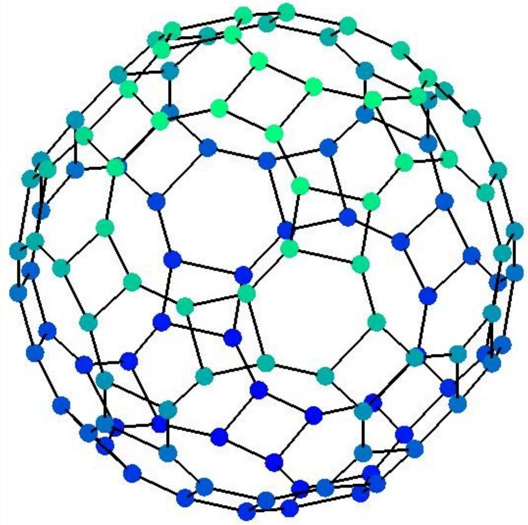
(the domain where $(\alpha \cdot x) > 0$).



The ground state function vanishes at the boundary.

It has a maximum at

$$x_1 = 0.387$$
$$x_2 = 2.941$$
$$x_3 = 0.446$$



The algebraic form of the Hamiltonian

Make a gauge rotation of the Hamiltonian:

$$h_{H_3} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_3} - E_0)(\Psi_0)$$

New spectral problem arises

$$h_{H_3}\phi(x) = -2\epsilon\phi(x)$$

with spectral parameter $\epsilon = E - E_0$

Can we find variables leading to an algebraic form of h_{H_3}?

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What might those variables be?

The invariants of the H_3 group

- Consider the fundamental weights of Δ_{H_3} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, \varphi_+, 1)$	12
$\omega_2 = (1, \varphi_+^2, 0)$	20
$\omega_3 = (0, 2\varphi_+, 0)$	30

- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega_1} (\omega \cdot x)^a$$

$a = 2, 6, 10$ are the degrees of the H_3 group

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$a = 2, 6, 10$ are the degrees of the H_3 group

- The invariants are defined ambiguously

$$
\begin{aligned}
t_2 &\longrightarrow t_2 \\
t_6 &\longrightarrow t_6 + \alpha_1 t_2^3 \\
t_{10} &\longrightarrow t_{10} + \alpha_2 t_2^2 t_6 + \alpha_3 t_2^5
\end{aligned}
$$

- We look for parameters α_i such that
  - the Hamiltonian h_{H_3} has algebraic form
  - has infinitely-many invariant subspaces in polynomials
  - these subspaces form a flag
  - the flag is “minimal”

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Those variables are

$$\tau_1 = \frac{1}{10+2\sqrt{5}}\ t_2$$
$$\tau_2 = \frac{1}{10+16\sqrt{5}}\left(t_6 - \frac{13}{10}t_2^3\right)$$
$$\tau_3 = \frac{1}{250+110\sqrt{5}}\left(t_{10} - \frac{76}{15}t_2^2 t_6 + \frac{1531}{375}t_2^5\right)$$

▸ Explicit expressions

The Hamiltonian takes the algebraic form

$$h_{H_3} = \sum_{i,j=1}^{3} A_{ij}\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{j=1}^{3} B_j\frac{\partial}{\partial\tau_j}$$

with

$$
\begin{aligned}
A_{11} &= 4\tau_1 \\
A_{12} &= 12\tau_2 \\
A_{13} &= 20\tau_3 \\
A_{22} &= -\frac{48}{5}\tau_1^2\tau_2 + \frac{45}{2}\tau_3 \\
A_{23} &= \frac{16}{15}\tau_1\tau_2^2 - 24\tau_1^2\tau_3 \\
A_{33} &= -\frac{64}{3}\tau_1\tau_2\tau_3 + \frac{128}{45}\tau_2^3 \\
B_1 &= 6 + 60\nu - 4\omega\tau_1 \\
B_2 &= -\frac{48}{5}(1+5\nu)\tau_1^2 - 12\omega\tau_2 \\
B_3 &= -\frac{64}{15}(2+5\nu)\tau_1\tau_2 - 20\omega\tau_3
\end{aligned}
$$

Configuration space and Jacobian

In τ's the configuration space boundary is an algebraic surface of degree 7 (degree 30 in x)

$$\kappa(\tau) = -12960\tau_1^5\tau_3^2 + 5760\tau_1^4\tau_2^2\tau_3 - 640\tau_1^3\tau_2^4 - 54000\tau_1^2\tau_2\tau_3^2 + 21600\tau_1\tau_2^3\tau_3 - 2304\tau_2^5 - 50625\tau_3^3 = 0$$

Boundary corresponds to zeros of Ψ_0

The square of the Jacobian of the change of variables $x \to \tau$ vanishes on this boundary:

$$J^2 = \begin{vmatrix} \frac{\partial\tau_1}{\partial x_1} & \frac{\partial\tau_1}{\partial x_2} & \frac{\partial\tau_1}{\partial x_3} \\ \frac{\partial\tau_2}{\partial x_1} & \frac{\partial\tau_2}{\partial x_2} & \frac{\partial\tau_2}{\partial x_3} \\ \frac{\partial\tau_3}{\partial x_1} & \frac{\partial\tau_3}{\partial x_2} & \frac{\partial\tau_3}{\partial x_3} \end{vmatrix}^2 = \frac{9}{5} \prod_{\alpha\in\mathcal{R}_3^+} (\alpha\cdot x)^2 = \frac{8}{45}\,\kappa(\tau)\,.$$

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Polynomial spaces

The algebraic operator h_{H_3} preserves subspaces

$$\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3} | 0 \le n_1 + 2n_2 + 3n_3 \le n \rangle \ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,2,3), they form an *infinite flag*

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n \subset \cdots$$

The flag is invariant with respect to weighted-projective transformations:

$$\tau_1 \longrightarrow \tau_1 + a$$
$$\tau_2 \longrightarrow \tau_2 + b_1\tau_1^2 + b_2\tau_1 + b_3$$
$$\tau_3 \longrightarrow \tau_3 + c_1\tau_1\tau_2 + c_2\tau_1^3 + c_3\tau_2 + c_4\tau_1^2 + c_5\tau_1 + c_6$$

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Eigenfunctions and spectrum

One can find the spectrum of h_{H_3} explicitly:

$$\epsilon_{k_1,k_2,k_3} = 2\omega(k_1 + 3k_2 + 5k_3) \ , \quad k_i = 0, 1, 2, \ldots$$

Degeneracy: $k_1 + 3k_2 + 5k_3 = \text{integer}$

The energies of the original Hamiltonian are

$$E = E_0 + \epsilon$$

Eigenfunctions $\phi_{n,i}$ of h_{H_3} are elements of $\mathcal{P}_n^{(1,2,3)}$.
Eigenfunctions of $\mathcal{H}$ are

$$\Psi = \Psi_0 \ \phi \qquad \text{(factorization)}$$

- $\mathbf{n = 0}$:

$$\phi_{0,0} = 1 \,, \qquad \epsilon_{0,0} = 0 \,.$$

- $\mathbf{n = 1}$:

$$\phi_{1,0} = \tau_1 + \frac{3}{2\omega}(1+10\nu) \,, \qquad \epsilon_{1,0} = 2\omega \,.$$

- $\mathbf{n = 2}$:

$$\begin{aligned} \phi_{2,0} &= \tau_1^2 - \frac{5}{\omega}(1+6\nu)\tau_1 + \frac{15}{4\omega^2}(1+6\nu)(1+10\nu) \,, \\ \epsilon_{2,0} &= 4\omega \,, \end{aligned}$$

$$\begin{aligned} \phi_{2,1} &= \tau_2 + \frac{12}{5\omega}(1+5\nu)\tau_1^2 - \frac{6}{\omega^2}(1+5\nu)(1+6\nu)\tau_1 \\ &\quad + \frac{3}{\omega^3}(1+5\nu)(1+6\nu)(1+10\nu) \,, \\ \epsilon_{2,1} &= 6\omega \,. \end{aligned}$$

The $h^{(3)}$ algebra.

Can $\mathcal{P}_n^{(1,2,3)}$ be finite-dimensional representation spaces of a Lie algebra of differential operators? Yes

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated (30 operators).

Generating elements can be split in two classes:

- First class: *lowering and Cartan operators*, they act on $\mathcal{P}_n$ at any n, infinite flag is preserved
- Second class: *raising operators*, a single space at a certain n is preserved

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First order operators

The first class generators consist of 13 first order operators

$$T_0^{(1)} = \partial_1 \,, \qquad T_0^{(2)} = \partial_2 \,, \qquad T_0^{(3)} = \partial_3 \,,$$
$$T_1^{(1)} = \tau_1\partial_1 \,, \qquad T_2^{(2)} = \tau_2\partial_2 \,, \qquad T_3^{(3)} = \tau_3\partial_3 \,,$$
$$T_1^{(3)} = \tau_1\partial_3 \,, \qquad T_{11}^{(3)} = \tau_1^2\partial_3 \,, \qquad T_{111}^{(3)} = \tau_1^3\partial_3 \,,$$
$$T_1^{(2)} = \tau_1\partial_2 \,, \qquad T_{11}^{(2)} = \tau_1^2\partial_2 \,, \qquad T_2^{(3)} = \tau_2\partial_3 \,,$$
$$T_{12}^{(3)} = \tau_1\tau_2\partial_3$$

Second and third order operators

plus 6 second order generators

$$T_2^{(11)} = \tau_2 \partial_{11} \,, \qquad T_{22}^{(13)} = \tau_2^2 \partial_{13} \,, \qquad T_{222}^{(33)} = \tau_2^3 \partial_{33} \,,$$
$$T_3^{(12)} = \tau_3 \partial_{12} \,, \qquad T_3^{(22)} = \tau_3 \partial_{22} \,, \qquad T_{13}^{(22)} = \tau_1 \tau_3 \partial_{22}$$

and 2 third order generators

$$T_3^{(111)} = \tau_3 \partial_{111} \,, \qquad T_{33}^{(222)} = \tau_3^2 \partial_{222}$$

These 21 operators are generating elements of the flag-preserving subalgebra of $h^{(3)}$

Second class (raising operators)

Define the auxiliary operator (which belongs to the first class)

$$J_0 = \tau_1 \partial_1 + 2\tau_2 \partial_2 + 3\tau_3 \partial_3 - n$$

Raising generators consist of 8 operators of 1st, 2nd and 3rd order

$$J_1^+ = \tau_1 J_0\,, \quad J_{2,-1}^+ = \tau_2 \partial_1 J_0\,, \quad J_{3,-2}^+ = \tau_3 \partial_2 J_0\,,$$
$$J_2^+ = \tau_2 J_0(J_0+1)\,, \quad J_{3,-11}^+ = \tau_3 \partial_{11} J_0\,, \quad J_{22,-3}^+ = \tau_2^2 \partial_3 J_0\,,$$
$$J_3^+ = \tau_3 J_0(J_0+1)(J_0+2)\,, \quad J_{3,-1}^+ = \tau_3 \partial_1 J_0(J_0+1)$$

$h^{(3)}$ is the infinite dimensional algebra of monomials in the 30 (22+8) generating elements

Subalgebras of $h^{(3)}$

Generating elements of $h^{(3)}$ can be grouped in 10 Abelian subalgebras

$$
\begin{aligned}
L = \{T_0^{(3)}, T_1^{(3)}, T_{11}^{(3)}, T_{111}^{(3)}\} &\longleftrightarrow \mathfrak{L} = \{T_3^{(111)}, J^+_{3,-11}, J^+_{3,-1}, J^+_3\} \\
R = \{T_0^{(2)}, T_1^{(2)}, T_{11}^{(2)}\} &\longleftrightarrow \mathfrak{R} = \{T_2^{(11)}, J^+_{2,-1}, J^+_2\} \\
F = \{T_2^{(3)}, T_{12}^{(3)}\} &\longleftrightarrow \mathfrak{F} = \{T_3^{(12)}, J^+_{3,-2}\} \\
E = \{T_{13}^{(22)}, T_3^{(22)}\} &\longleftrightarrow \mathfrak{E} = \{T_{22}^{(13)}, J^+_{22,-3}\} \\
G = \{T_{222}^{(33)}\} &\longleftrightarrow \mathfrak{G} = \{T_{33}^{(222)}\}
\end{aligned}
$$

and a closed subalgebra

$$B = \{T_0^{(1)}, T_1^{(1)}, T_2^{(2)}, T_3^{(3)}, J_0, J_1^+\}$$

Commutation relations between commutative algebras:

$$
\begin{aligned}
[L, R] &= 0, & [\mathfrak{L}, \mathfrak{R}] &= 0,\\
[L, F] &= 0, & [\mathfrak{L}, \mathfrak{F}] &= 0,\\
[L, E] &= P_2(R), & [\mathfrak{L}, \mathfrak{E}] &= P_2(\mathfrak{R}),\\
[L, G] &= 0, & [\mathfrak{L}, \mathfrak{G}] &= 0,\\
[R, F] &= L, & [\mathfrak{R}, \mathfrak{F}] &= \mathfrak{L},\\
[R, E] &= 0, & [\mathfrak{R}, \mathfrak{E}] &= 0,\\
[R, G] &= P_2(F), & [\mathfrak{R}, \mathfrak{G}] &= P_2(\mathfrak{F}),\\
[F, E] &= P_2(R \oplus B), & [\mathfrak{F}, \mathfrak{E}] &= P_2(\mathfrak{R} \oplus B),\\
[F, G] &= 0, & [\mathfrak{F}, \mathfrak{G}] &= 0,\\
[E, G] &= P_3(F \oplus B), & [\mathfrak{E}, \mathfrak{G}] &= P_3(\mathfrak{F} \oplus B),
\end{aligned}
$$

$$[L, \mathfrak{R}] = P_2(F \oplus B), \qquad [\mathfrak{L}, R] = P_2(\mathfrak{F} \oplus B),$$
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$$[R, \mathfrak{F}] = E, \qquad [\mathfrak{R}, F] = \mathfrak{E},$$
$$[R, \mathfrak{E}] = P_2(F \oplus B), \qquad [\mathfrak{R}, E] = P_2(\mathfrak{F} \oplus B),$$
$$[R, \mathfrak{G}] = 0, \qquad [\mathfrak{R}, G] = 0,$$
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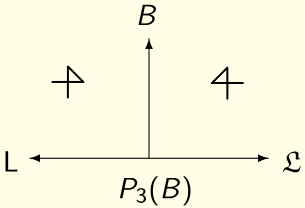
$$[E, \mathfrak{G}] = 0, \qquad [\mathfrak{E}, G] = 0,$$

$$[L,\mathfrak{L}] = P_3(B), \qquad [R,\mathfrak{R}] = P_2(B), \qquad [F,\mathfrak{F}] = P_2(B),$$
$$[E,\mathfrak{E}] = P_3(B), \qquad [G,\mathfrak{G}] = P_4(B)$$

Commutation relations between Abelian subalgebras and B:

$$[L,B] = L\,, \quad [R,B] = R\,, \quad [F,B] = F\,, \quad [E,B] = E\,, \quad [G,B] = G\,,$$
$$[\mathfrak{L},B] = \mathfrak{L}\,, \quad [\mathfrak{R},B] = \mathfrak{R}\,, \quad [\mathfrak{F},B] = \mathfrak{F}\,, \quad [\mathfrak{E},B] = \mathfrak{E}\,, \quad [\mathfrak{G},B] = \mathfrak{G}$$



Commutation relations between generators of B:

$$
\begin{array}{lll}
[T_0^{(1)}, T_1^{(1)}] = T_0^{(1)} , & [T_0^{(1)}, T_2^{(2)}] = 0 , & [T_0^{(1)}, T_3^{(3)}] = 0 , \\
[T_0^{(1)}, J_0] = T_0^{(1)} , & [T_0^{(1)}, J_1^{+}] = T_1^{(1)} + J_0 , & [T_1^{(1)}, T_2^{(2)}] = 0 , \\
[T_1^{(1)}, T_3^{(3)}] = 0 , & [T_1^{(1)}, J_0] = 0 , & [T_1^{(1)}, J_1^{+}] = J_1^{+} , \\
[T_2^{(2)}, T_3^{(3)}] = 0 , & [T_2^{(2)}, J_0] = 0 , & [T_2^{(2)}, J_1^{+}] = 0 , \\
[T_3^{(3)}, J_0] = 0 , & [T_3^{(3)}, J_1^{+}] = 0 , & [J_0, J_1^{+}] = J_1^{+}
\end{array}
$$

Correspond to

$$B \cong g\ell_2 \oplus \mathcal{R}^{(2)}$$

The h_{H_3} Hamiltonian (Lie algebraic form)

Lie algebraic form for h_{H_3}:

$$
\begin{aligned}
h_{H_3} = {} & 4\,T_1^{(1)}\,T_0^{(1)} + 24\,T_2^{(2)}\,T_0^{(1)} + 40\,T_3^{(3)}\,T_0^{(1)} - \frac{48}{5}\,T_2^{(2)}\,T_{11}^{(2)} \\
& + \frac{45}{2}\,T_3^{(22)} + \frac{32}{15}\,T_{12}^{(3)}\,T_2^{(2)} - 48\,T_3^{(3)}\,T_{11}^{(2)} - \frac{64}{3}\,T_3^{(3)}\,T_{12}^{(3)} \\
& + \frac{128}{45}\,T_{222}^{(33)} + (6+60\nu)\,T_0^{(1)} - 4\omega\,T_1^{(1)} - \frac{48}{5}(1+5\nu)\,T_{11}^{(2)} \\
& - 12\omega\,T_2^{(2)} - \frac{64}{15}(2+5\nu)\,T_{12}^{(3)} - 20\omega\,T_3^{(3)}
\end{aligned}
$$

The H_4 integrable model

The H_4 rational Hamiltonian is

$$\mathcal{H}_{H_4} = \frac{1}{2}\sum_{k=1}^{4}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{g}{x_k^2}\right] + \sum_{\mu_{2,3,4}=0,1} \frac{2g}{[x_1 + (-1)^{\mu_2} x_2 + (-1)^{\mu_3} x_3 + (-1)^{\mu_4} x_4]^2} + \sum_{\{i,j,k,l\}} \sum_{\mu_{1,2}=0,1} \frac{2g}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k + 0\cdot x_l]^2},$$

where $\{i,j,k,l\} = \{1,2,3,4\}$ and all even permutations. The coupling constant is

$$g = \nu(\nu-1) > -\frac{1}{4}\,, \quad \text{and} \quad \varphi_\pm = \frac{1\pm\sqrt{5}}{2}$$

The Hamiltonian is invariant wrt the H_4 Coxeter group, which is the symmetry group of the *600-cell*. This group is a subgroup of $O(4)$ and has order 14400

The Hamiltonian is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j$$

$$\varphi_+ \longleftrightarrow \varphi_-$$

Configuration space is the domain in $\mathbf{R}^4$ where $(\alpha \cdot x) > 0$ with $\alpha > 0$.

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The ground state function and its eigenvalue are

$$\Psi_0 = \Delta_1^\nu \Delta_2^\nu \Delta_3^\nu \exp\left(-\frac{\omega}{2}\sum_{k=1}^{4} x_k^2\right) \ , \quad E_0 = 2\omega(1+30\nu)$$

where

$$\Delta_1 = \prod_{k=1}^{4} x_k,$$

$$\Delta_2 = \prod_{\mu_{2,3,4}=0,1} [x_1 + (-1)^{\mu_2} x_2 + (-1)^{\mu_3} x_3 + (-1)^{\mu_4} x_4],$$

$$\Delta_3 = \prod_{\{i,j,k,l\}} \prod_{\mu_{1,2}=0,1} [x_i + (-1)^{\mu_1} \varphi_+ x_j + (-1)^{\mu_2} \varphi_- x_k + 0 \cdot x_l]\,.$$

The algebraic form of the Hamiltonian

To obtain the algebraic form one proceeds analogously to H_3:

- Make a gauge rotation of the Hamiltonian:

$$h_{H_4} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_4} - E_0)(\Psi_0) \ .$$

- Consider the fundamental weights of Δ_{H_4} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, 0, 0, 2\varphi_+)$	120
$\omega_2 = (1, \varphi_+^2, 0, \varphi_+^4)$	600
$\omega_3 = (0, \varphi_+, 1, \varphi_+^4 - 1)$	720
$\omega_4 = (0, 2\varphi_+, 0, 2\varphi_+^3)$	1200

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- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega} (\omega \cdot x)^a$$

$a = 2, 12, 20, 30$ are the degrees of the H_4 group.

- The invariants are defined ambiguously

$$t_2 \longrightarrow t_2$$
$$t_{12} \longrightarrow t_{12} + \alpha_1 t_2^6$$
$$t_{20} \longrightarrow t_{20} + \alpha_2 t_2^4 t_{12} + \alpha_3 t_2^{10}$$
$$t_{30} \longrightarrow t_{30} + \alpha_4 t_2^5 t_{20} + \alpha_5 t_2^3 t_{12}^2 + \alpha_6 t_2^9 t_{12} + \alpha_7 t_2^{15}$$

- We look for parameters α_i such that
  - the Hamiltonian h_{H_4} has an algebraic form
  - has infinitely-many invariant subspaces in polynomials
  - these subspaces form a flag
  - the flag is minimal

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Those variables are

$$\tau_1 = \frac{1}{60(3+\sqrt{5})}\; t_2$$

$$\tau_2 = -\frac{1}{1680(161+72\sqrt{5})}\left(t_{12} - t_2^6\right)$$

$$\tau_3 = \frac{1}{29920(15127+6765\sqrt{5})}\left(t_{20} - \frac{43510}{1809}t_2^4 t_{12} + \frac{41701}{1809}t_2^{10}\right)$$

$$\tau_4 = \frac{1}{480480(930249+416020\sqrt{5})}\left(t_{30} - \frac{17583778485}{146142376}t_2^5 t_{20}\right.$$
$$\left. - \frac{313009515}{15383408}t_2^3 t_{12}^2 + \frac{22081114965}{7691704}t_2^9 t_{12} - \frac{798259915667}{292284752}t_2^{15}\right)$$

The Hamiltonian takes the algebraic form

$$h_{H_4} = \sum_{i,j=1}^{4} A_{ij} \frac{\partial^2}{\partial\tau_i \partial\tau_j} + \sum_{j=1}^{4} B_j \frac{\partial}{\partial\tau_j}$$

with

$$\begin{aligned}
A_{11} &= 4\ \tau_1\ , \\
A_{12} &= 24\ \tau_2\ , \\
A_{13} &= 40\ \tau_3\ , \\
A_{14} &= 60\ \tau_4\ , \\
A_{22} &= 88\ \tau_1\tau_3 + 8\ \tau_1^5\tau_2\ , \\
A_{23} &= -4\ \tau_1^3\tau_2^2 + 24\ \tau_1^5\tau_3 - 8\ \tau_4\ ,
\end{aligned}$$

$$A_{24} = 10\ \tau_1^2\tau_2^3 + 60\ \tau_1^4\tau_2\tau_3 + 40\ \tau_1^5\tau_4 - 600\ \tau_3^2\ ,$$

$$A_{33} = -\frac{38}{3}\ \tau_1\tau_2^3 + 28\ \tau_1^3\tau_2\tau_3 - \frac{8}{3}\ \tau_1^4\tau_4\ ,$$

$$A_{34} = 210\ \tau_1^2\tau_2^2\tau_3 + 60\ \tau_1^3\tau_2\tau_4 - 180\ \tau_1^4\tau_3^2 + 30\ \tau_2^4\ ,$$

$$A_{44} = -2175\ \tau_1\tau_2^3\tau_3 - 450\tau_1^2\tau_2^2\tau_4 - 1350\ \tau_1^3\tau_2\tau_3^2 - 600\ \tau_1^4\tau_3\tau_4\ ,$$

$$B_1 = 8 + 240\nu - 4\omega\tau_1\ ,$$

$$B_2 = 12(1 + 10\nu)\ \tau_1^5 - 24\omega\tau_2\ ,$$

$$B_3 = 20(1 + 6\nu)\ \tau_1^3\tau_2 - 40\omega\tau_3\ ,$$

$$B_4 = 15(1 - 30\nu)\ \tau_1^2\tau_2^2 - 450(1 + 2\nu)\ \tau_1^4\tau_3 - 60\omega\tau_4\ .$$

Configuration space

In τ's the configuration space boundary is an algebraic surface of degree 18 (degree 120 in x)

$$
\begin{aligned}
&64\ \tau_1^{15}\tau_4^3 + 1440\ \tau_1^{14}\tau_2\tau_3\tau_4^2 + 10800\ \tau_1^{13}\tau_2^2\tau_3^2\tau_4 + 27000\ \tau_1^{12}\tau_2^3\tau_3^3\\
&- 240\ \tau_1^{12}\tau_2^3\tau_4^2 - 3600\ \tau_1^{11}\tau_2^4\tau_3\tau_4 - 13500\ \tau_1^{10}\tau_2^5\tau_3^2 + 34992\ \tau_1^{10}\tau_3^5\\
&- 1440\ \tau_1^{10}\tau_3^2\tau_4^2 + 300\ \tau_1^9\tau_2^6\tau_4 - 2160\ \tau_1^9\tau_2\tau_3^3\tau_4 - 1440\ \tau_1^9\tau_2\tau_4^3\\
&+ 2250\ \tau_1^8\tau_2^7\tau_3 - 22680\ \tau_1^8\tau_2^2\tau_3^4 - 28080\ \tau_1^8\tau_2^2\tau_3\tau_4^2 - 203760\ \tau_1^7\tau_2^3\tau_3^2\tau_4\\
&- 125\ \tau_1^6\tau_2^9 - 493020\ \tau_1^6\tau_2^4\tau_3^3 + 3600\ \tau_1^6\tau_2^4\tau_4^2 + 57780\ \tau_1^5\tau_2^5\tau_3\tau_4\\
&- 8640\ \tau_1^5\tau_3^4\tau_4 + 4320\ \tau_1^5\tau_3\tau_4^3 + 221310\ \tau_1^4\tau_2^6\tau_3^2 - 648000\ \tau_1^4\tau_2\tau_3^5\\
&+ 116640\ \tau_1^4\tau_2\tau_3^2\tau_4^2 - 4680\ \tau_1^3\tau_2^7\tau_4 + 712800\ \tau_1^3\tau_2^2\tau_3^3\tau_4 + 6480\ \tau_1^3\tau_2^2\tau_4^3\\
&- 35640\ \tau_1^2\tau_2^8\tau_3 + 2052000\ \tau_1^2\tau_2^3\tau_3^4 + 62640\ \tau_1^2\tau_2^3\tau_3\tau_4^2 + 259200\ \tau_1\tau_2^4\tau_3^2\tau_4\\
&+ 1944\ \tau_2^{10} + 129600\ \tau_2^5\tau_3^3 + 2592\ \tau_2^5\tau_4^2 + 2160000\ \tau_3^6 - 86400\ \tau_3^3\tau_4^2\\
&+ 864\ \tau_4^4 = 0
\end{aligned}
$$

Invariant spaces

The algebraic operator h_{H_4} preserves subspaces

$$\mathcal{P}_n^{(1,5,8,12)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3}\tau_4^{n_4} | 0 \le n_1 + 5n_2 + 8n_3 + 12n_4 \le n \rangle \ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,5,8,12), they form flag

The flag is invariant with respect to weighted-projective transformations:

$$\begin{aligned}
\tau_1 &\to \tau_1 + a\,, \\
\tau_2 &\to \tau_2 + b_1\,\tau_1^5 + b_2\,\tau_1^4 + b_3\,\tau_1^3 + b_4\,\tau_1^2 + b_5\,\tau_1 + b_6\,, \\
\tau_3 &\to \tau_3 + c_1\,\tau_1^3\tau_2 + c_2\,\tau_1^2\tau_2 + c_3\,\tau_1\tau_2 + c_4\,\tau_2 + c_5\,\tau_1^8 + c_6\,\tau_1^7 \\
&\quad + c_7\,\tau_1^6 + c_8\,\tau_1^5 + c_9\,\tau_1^4 + c_{10}\,\tau_1^3 + c_{11}\,\tau_1^2 + c_{12}\,\tau_1 + c_{13}\,, \\
\tau_4 &\to \tau_4 + d_1\,\tau_1^4\tau_3 + d_2\,\tau_1^3\tau_3 + d_3\,\tau_1^2\tau_3 + d_4\,\tau_1\tau_3 + d_5\,\tau_3 \\
&\quad + d_6\,\tau_1^7\tau_2 + d_7\,\tau_1^6\tau_2 + d_8\,\tau_1^5\tau_2 + d_9\,\tau_1^4\tau_2 + d_{10}\,\tau_1^3\tau_2 \\
&\quad + d_{11}\,\tau_1^2\tau_2 + d_{12}\,\tau_1\tau_2 + d_{13}\,\tau_2 + d_{14}\,\tau_1^{12} + d_{15}\,\tau_1^{11} \\
&\quad + d_{16}\,\tau_1^{10} + d_{17}\,\tau_1^9 + d_{18}\,\tau_1^8 + d_{19}\,\tau_1^7 + d_{20}\,\tau_1^6 + d_{21}\,\tau_1^5 \\
&\quad + d_{22}\,\tau_1^4 + d_{23}\,\tau_1^3 + d_{24}\,\tau_1^2 + d_{25}\,\tau_1 + d_{26}\,,
\end{aligned}$$

Eigenfunctions and spectrum

Spectrum of h_{H_4}:

$$\epsilon_{k_1,k_2,k_3,k_4} = 2\omega(k_1 + 6k_2 + 10k_3 + 15k_4)\ , \quad k_i = 0, 1, 2, \ldots$$

Degeneracy: $\quad k_1 + 6k_2 + 10k_3 + 15k_4 = \text{integer}$

The energies of the original Hamiltonian are

$$E = E_0 + \epsilon$$

Eigenfunctions $\phi_{n,i}$ of h_{H_4} are elements of $\mathcal{P}_n^{(1,5,8,12)}$.
Eigenfunctions of $\mathcal{H}_{H_4}$ are

$$\Psi = \Psi_0\ \phi \qquad \text{(factorization)}$$

- $\mathbf{n = 0}$:

$$\phi_{0,0} = 1 \ , \qquad \epsilon_{0,0} = 0 \ .$$

- $\mathbf{n = 1}$:

$$\phi_{1,0} = \tau_1 - \frac{2}{\omega}(1 + 30\nu) \ , \qquad \epsilon_{1,0} = 2\omega \ .$$

- $\mathbf{n = 2}$:

$$\phi_{2,0} = \tau_1^2 - \frac{6}{\omega}(1+20\nu)\tau_1 + \frac{6}{\omega^2}(1+20\nu)(1+30\nu) \ , \qquad \epsilon_{2,0} = 4\omega \ .$$

Conclusion

- Algebraic forms for the H_3 and H_4 rational model exist. They act on the spaces of polynomials $\mathcal{P}_n^{(1,2,3)}$ and $\mathcal{P}_n^{(1,5,8,12)}$. Eigenfunctions are elements of the respective spaces.
- The hidden algebra of the H_3 model is the $h^{(3)}$ algebra, which has infinite dimension but is finitely generated
- It is possible to construct an isospectral discrete model and a quasi-exactly-solvable generalization for both models.
- An integral of motion exists for each model exists. It has an algebraic form in τ variables. Other integral(s) of motion has not been found yet.

$$\tau_1 = x_1^2 + x_2^2 + x_3^2 \,,$$

$$\begin{aligned}\tau_2 = & -\frac{3}{10}\,(x_1^6 + x_2^6 + x_3^6) + \frac{3}{10}(2 - 5\varphi_+)\,(x_1^2 x_2^4 + x_2^2 x_3^4 + x_3^2 x_1^4) \\ & + \frac{3}{10}(2 - 5\varphi_-)\,(x_1^2 x_3^4 + x_2^2 x_1^4 + x_3^2 x_2^4) - \frac{39}{5}\,(x_1^2 x_2^2 x_3^2)\,,\end{aligned}$$

$$\begin{aligned}\tau_3 = {} & \frac{2}{125}\left(x_1^{10}+x_2^{10}+x_3^{10}\right)+\frac{2}{25}(1+5\varphi_-)\left(x_1^8x_2^2+x_2^8x_3^2+x_3^8x_1^2\right)\\ & +\frac{2}{25}(1+5\varphi_+)\left(x_1^8x_3^2+x_2^8x_1^2+x_3^8x_2^2\right)\\ & +\frac{4}{25}(1-5\varphi_-)\left(x_1^6x_2^4+x_2^6x_3^4+x_3^6x_1^4\right)\\ & +\frac{4}{25}(1-5\varphi_+)\left(x_1^6x_3^4+x_2^6x_1^4+x_3^6x_2^4\right)\\ & -\frac{112}{25}\left(x_1^6x_2^2x_3^2+x_2^6x_3^2x_1^2+x_3^6x_1^2x_2^2\right)\\ & +\frac{212}{25}\left(x_1^2x_2^4x_3^4+x_2^2x_3^4x_1^4+x_3^2x_1^4x_2^4\right).\end{aligned}$$

Back to presentation