Toward a stochastic description of reheating

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Based on 1705.????, with

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H. Xie (U. Wisconsin)

RICE

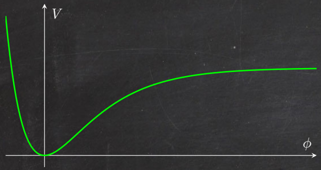
Congratulations!

(and thank you for the invitation)

- Motivation
- Stochastic Particle Production
- Exact Results
- Conclusion

Motivation

All you need to drive inflation ($a \sim e^{Ht}$) is a scalar field,



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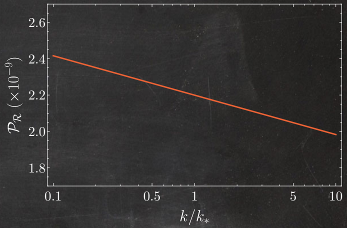

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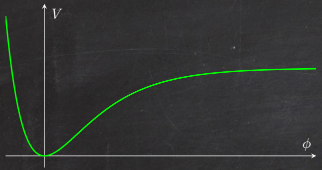

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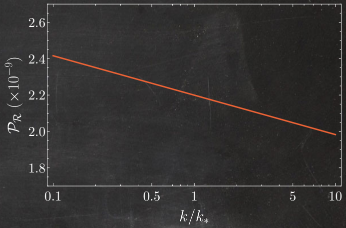
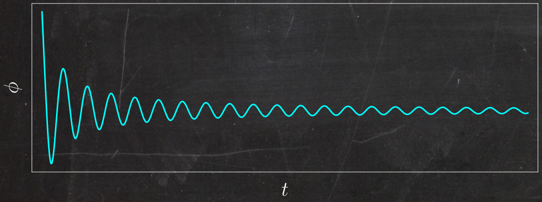


When inflation ends, the Universe reheats…



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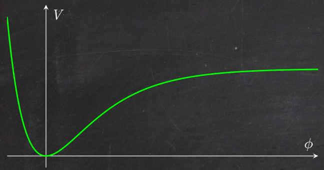

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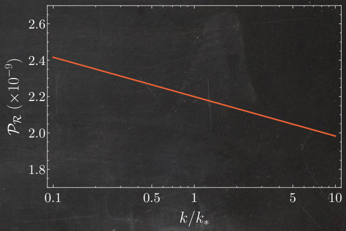

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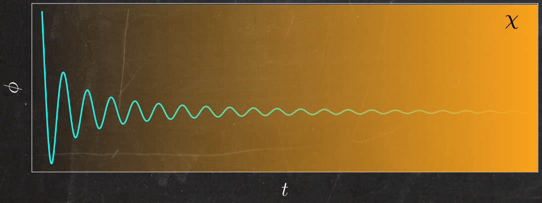

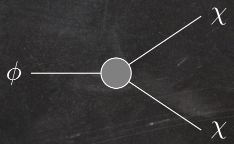

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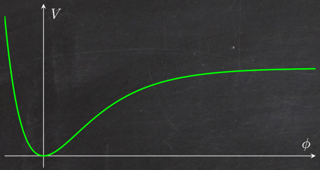

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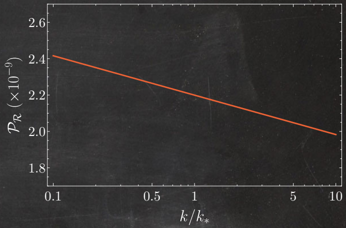
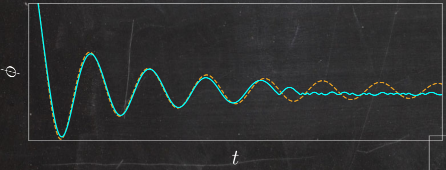
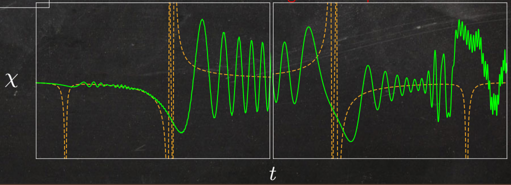


When inflation ends, the Universe reheats…in a potentially complicated way



Ellis, G, Nagata, Nanopoulos, Olive 2017



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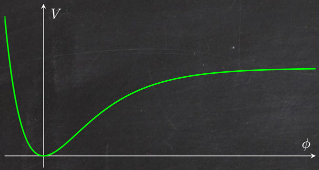

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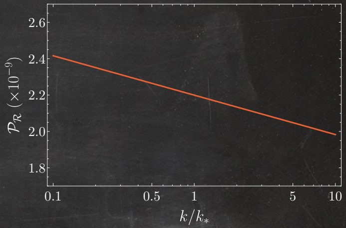

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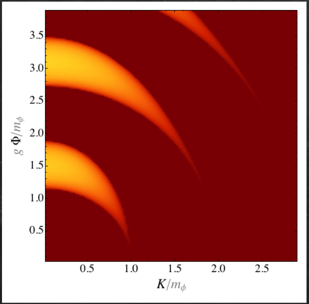
Amin, Hertzberg, Kaiser, Karouby 2015



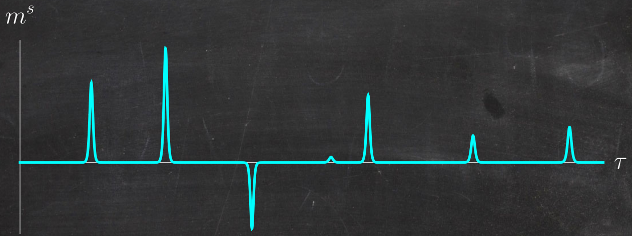
$$\left[\mathbb{1}\left(\partial_\tau^2 + (k/a)^2\right) + \mathbf{p}(\tau)\partial_\tau + \mathbf{m}(\tau)\right]\cdot\delta\boldsymbol{\chi} = 0$$

$$(\mathbf{p})^a_{\ b} = 2\Gamma^a_{\ bc}\phi^{c\prime} + 3H\delta^a_b$$
$$(\mathbf{m})^a_{\ b} = (G^{ac}V_{,c})_{,b} + \Gamma^a_{\ cd,b}\phi^{c\prime}\phi^{d\prime}$$

Stochastic Particle Production

Consider N_f coupled (scalar) fields. Assume the evolution of fluctuations contains localized non-adiabatic events with random strengths at random intervals, and that the fields are otherwise free

$$\left[\mathbb{1}\,\partial_\tau^2 + \boldsymbol{\omega}^2 + \mathbf{m}^s(\tau)\right]\cdot\boldsymbol{\chi}(\tau,\mathbf{k}) = 0\,, \qquad \omega_a^2 = k^2 + m_a^2$$



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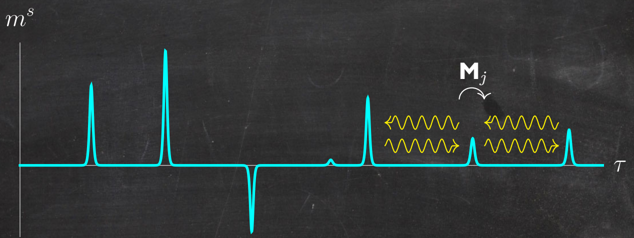

After the j-th event,

$$\chi_j^a(\tau) \equiv \frac{1}{\sqrt{2\omega_a}}\left[\beta_j^a e^{i\omega_a\tau} + \alpha_j^a e^{-i\omega_a\tau}\right]\,, \qquad \begin{pmatrix}\boldsymbol{\beta}_j\\ \boldsymbol{\alpha}_j\end{pmatrix} = \mathbf{M}_j\begin{pmatrix}\boldsymbol{\beta}_{j-1}\\ \boldsymbol{\alpha}_{j-1}\end{pmatrix}$$

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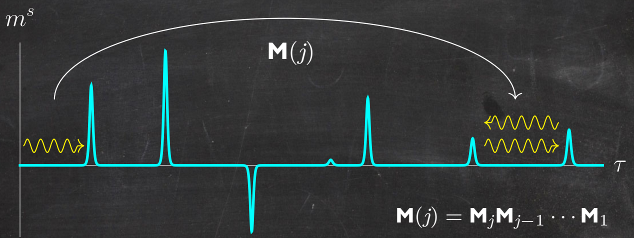

$$\mathbf{M}(j) = \mathbf{M}_j\mathbf{M}_{j-1}\cdots\mathbf{M}_1$$

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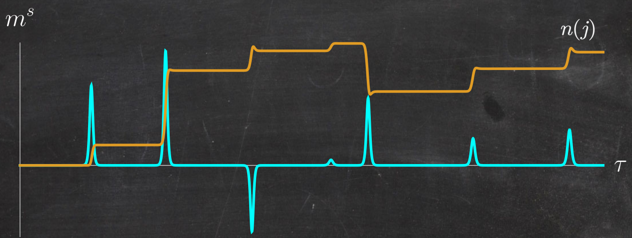
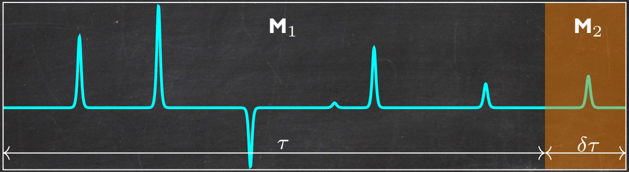


A random walk (with drift) for the occupation number

$$n_a(j) = \frac{1}{2\omega_a}\left(|\dot{\chi}_j^a|^2 + \omega_a^2|\chi_j^a|^2\right) - \frac{1}{2} = |\beta_j^a|^2$$

Randomness and non-adiabaticity are encoded in $\mathbf{M}$.
A probability distribution can be defined, $P(\mathbf{M};\tau)$



If the processes in the τ and $\delta\tau$ strips are uncorrelated, the distribution for $\mathbf{M}\equiv\mathbf{M}_2\mathbf{M}_1$ satisfies

$$P(\mathbf{M};\tau+\delta\tau)=\int d\mu(\mathbf{M}_2)\,P(\mathbf{M}_2^{-1}\mathbf{M};\tau)P(\mathbf{M}_2;\delta\tau)$$

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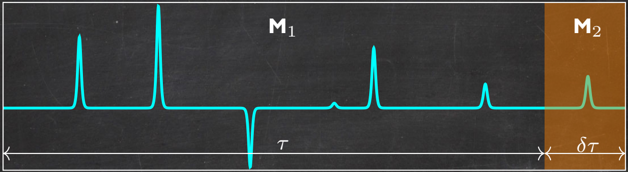

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$\Rightarrow$ Markovian evolution. Moreover, as $\delta\tau/\tau\longrightarrow 0$,

$$\partial_\tau P(\mathbf{M};\tau)=-\partial_{\mathbf{M}}\left[\frac{\langle\delta\mathbf{M}\rangle_{\mathbf{M}_2}}{\delta\tau}P(\mathbf{M};\tau)\right]+\frac{1}{2!}\partial^2_{\mathbf{M}}\left[\frac{\langle\delta\mathbf{M}^2\rangle_{\mathbf{M}_2}}{\delta\tau}P(\mathbf{M};\tau)\right]+\ldots$$

(Fokker-Planck)

A general transfer matrix can be parametrized as

$$\mathbf{M} = \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u}^* \end{pmatrix} \begin{pmatrix} \sqrt{1+\mathbf{n}} & \sqrt{\mathbf{n}} \\ \sqrt{\mathbf{n}} & \sqrt{1+\mathbf{n}} \end{pmatrix} \begin{pmatrix} \mathbf{v} & 0 \\ 0 & \mathbf{v}^* \end{pmatrix}$$

where $\mathbf{u}, \mathbf{v} \in U(N_{\rm f})$, and $\mathbf{n} = \mathrm{diag}(n_1, n_2, \cdots) \quad \Rightarrow \quad N_{\rm f}(2N_{\rm f}+1)$ variables in FP equation!

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Late-time equilibrium $\Rightarrow$ maximal entropy

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The Maximum Entropy Ansatz

Assume the building block P maximizes the Shannon entropy

$$S[P] = -\int P(\mathbf{M}_2; \delta\tau) \ln P(\mathbf{M}_2; \delta\tau)\, d\mu(\mathbf{M}_2)$$

subject to the constraints:

- The local mean particle production rate is known and fixed, $\mu_j \equiv \dfrac{1}{N_\mathrm{f}} \dfrac{\langle n_j \rangle_{\delta\tau}}{\delta\tau}$
- Coarse-grained continuity, $\mathbf{M}_{\tau+\delta\tau} \xrightarrow{\delta\tau \to 0} \mathbf{M}_\tau$

Consequences (Mello, Pereyra, Kumar 1988; Amin, Baumann 2016):

- P is independent of $\mathbf{u}$,

$$dP(\{\mathbf{u},\mathbf{n},\mathbf{v}\}) = P(\{\mathbf{n},\mathbf{v}\})\, d\mu(\mathbf{u})$$

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2. The marginal distribution $\int P(\{\mathbf{n},\mathbf{v}\})\, d\mu(\mathbf{v})$ satisfies the Fokker-Planck equation

$$\frac{1}{\mu}\frac{\partial}{\partial\tau}P(n_a;\tau) = \sum_{a=1}^{N_{\rm f}}\left[(1+2n_a)+\frac{1}{N_{\rm f}+1}\sum_{b\neq a}\frac{n_a+n_b+2n_an_b}{n_a-n_b}\right]\frac{\partial P}{\partial n_a}+\frac{2}{N_{\rm f}+1}\sum_{a=1}^{N_{\rm f}}n_a(1+n_a)\frac{\partial^2 P}{\partial n_a^2}$$

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3. A closed set of equations for the moments of $n=\sum_a n_a$ can be obtained. It implies

$$\partial_\tau\langle\ln(1+n)\rangle \xrightarrow{\tau\to\infty} \frac{2N_{\rm f}}{N_{\rm f}+1}\mu$$

$$\partial_\tau \mathrm{Var}[\ln(1+n)] \xrightarrow{\tau\to\infty} \frac{4}{N_{\rm f}+1}\mu$$

i.e. exponential growth for the occupation number

Exact Results

Consider the approximation

$$m^{\mathrm{s}}_{ab}(\tau) = 2\sqrt{\omega_a\omega_b}\sum_{j=1}^{N_{\mathrm{s}}}\Lambda_{ab}(\tau_j)\delta(\tau-\tau_j)\,,$$

where τ_j are uniformly distributed, and

$$\langle\Lambda_{ab}\rangle = 0, \qquad \langle\Lambda_{ab}\Lambda_{cd}\rangle = \sigma^2_{ab}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})$$

The transfer matrix takes the form

$$\mathbf{M}_j = \mathbb{1} + i\underbrace{\begin{pmatrix}\mathbf{a}_j^* & 0\\ 0 & \mathbf{a}_j\end{pmatrix}\begin{pmatrix}\mathbf{\Lambda}_j & \mathbf{\Lambda}_j\\ -\mathbf{\Lambda}_j & -\mathbf{\Lambda}_j\end{pmatrix}\begin{pmatrix}\mathbf{a}_j & 0\\ 0 & \mathbf{a}_j^*\end{pmatrix}}_{\mathbf{m}_j}, \qquad \mathbf{a}_j \equiv \mathrm{diag}(e^{i\omega_1\tau_j}, e^{i\omega_2\tau_j}, \cdots)$$

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Will focus on the total occupation number. Define $\mathbf{R} = \mathbf{M}\mathbf{M}^\dagger$:

$$n(j) \;=\; \frac{1}{4}\mathrm{Tr}\left[\mathbf{M}(j)\mathbf{M}^\dagger(j) - \mathbb{1}\right] \;\equiv\; \frac{1}{4}\mathrm{Tr}\left[\mathbf{R}(j) - \mathbb{1}\right]$$

Algorithm:

1. Parametrize $\mathbf{R}$

$$\mathbf{R} = \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u}^* \end{pmatrix} \begin{pmatrix} \mathbf{f} & \tilde{\mathbf{f}} \\ \tilde{\mathbf{f}} & \mathbf{f} \end{pmatrix} \begin{pmatrix} \mathbf{u}^\dagger & 0 \\ 0 & \mathbf{u}^\top \end{pmatrix}, \qquad \begin{aligned} \mathbf{f} &= 2\mathbf{n} + \mathbb{1} \\ \tilde{\mathbf{f}} &= \sqrt{\mathbf{f}^2 - \mathbb{1}} \end{aligned}$$

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2. Solve for first and second order perturbations in terms of previous value of $\mathbf{R}$

$$\mathbf{R}(j+1) = \mathbf{R}(j) + \delta\mathbf{R}\,,$$

$$\delta\mathbf{R} = \mathbf{R}(j)\mathbf{m}^\dagger_{j+1} + \mathrm{h.c.} + \mathbf{m}_{j+1}\mathbf{R}(j)\mathbf{m}^\dagger_{j+1}\,.$$

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3. Calculate correlators

$$\langle \delta f_a^{(1)} \delta f_b^{(1)} \rangle_{\delta\tau} = 2\tilde{f}_a\tilde{f}_b \sum_{c,d} \sigma_{cd}^2 \left(u_{ac}^\dagger u_{ad}^\dagger u_{cb} u_{db} + u_{bc}^\dagger u_{bd}^\dagger u_{ca} u_{da} \right)$$

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4. Write and solve the FP equation

Single field

Only two parameters, f and $u = e^{i\phi}$. Computation is straightforward,

$$\langle \delta f^{(1)} \delta f^{(1)} \rangle = 2\tilde{f}^2 \sigma^2$$
$$\langle \delta f^{(1)} \delta \phi^{(1)} \rangle = 0$$
$$\langle \delta \phi^{(1)} \delta \phi^{(1)} \rangle = \frac{\sigma^2}{2\tilde{f}^2}(2\tilde{f}^2 + f^2)$$

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No ϕ dependence! $\Rightarrow$ maximum entropy

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No ϕ dependence! $\Rightarrow$ maximum entropy

The FP equation is

$$\frac{1}{\sigma^2}\frac{\partial}{\partial \tau}P(f;\tau) = \frac{\partial}{\partial f}\left[(f^2-1)\frac{\partial}{\partial f}P(f;\tau)\right]$$

Single field

Only two parameters, f and $u = e^{i\phi}$. Computation is straightforward,

$$\langle \delta f^{(1)} \delta f^{(1)} \rangle = 2\tilde{f}^2 \sigma^2$$

$$\langle \delta f^{(1)} \delta \phi^{(1)} \rangle = 0$$

$$\langle \delta \phi^{(1)} \delta \phi^{(1)} \rangle = \frac{\sigma^2}{2\tilde{f}^2}(2\tilde{f}^2 + f^2)$$

$$\langle \delta f^{(2)} \rangle = 2f\sigma^2$$

$$\langle \delta \phi^{(2)} \rangle = 0$$

No ϕ dependence! $\Rightarrow$ maximum entropy

The FP equation is

$$\frac{1}{\sigma^2}\frac{\partial}{\partial \tau}P(n;\tau) = \frac{\partial}{\partial n}\left[n^2 \frac{\partial}{\partial n}P(n;\tau)\right]$$

Single field

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$$\frac{1}{\sigma^2}\frac{\partial}{\partial \tau} P(n;\tau) = \frac{\partial}{\partial n}\left[n^2 \frac{\partial}{\partial n} P(n;\tau)\right]$$

with solution

$$P(n;\tau)\, dn = \frac{1}{\sqrt{4\pi\sigma^2\tau}} \exp\left[-\frac{(\ln n - \sigma^2\tau)^2}{4\sigma^2\tau}\right] d\ln n$$

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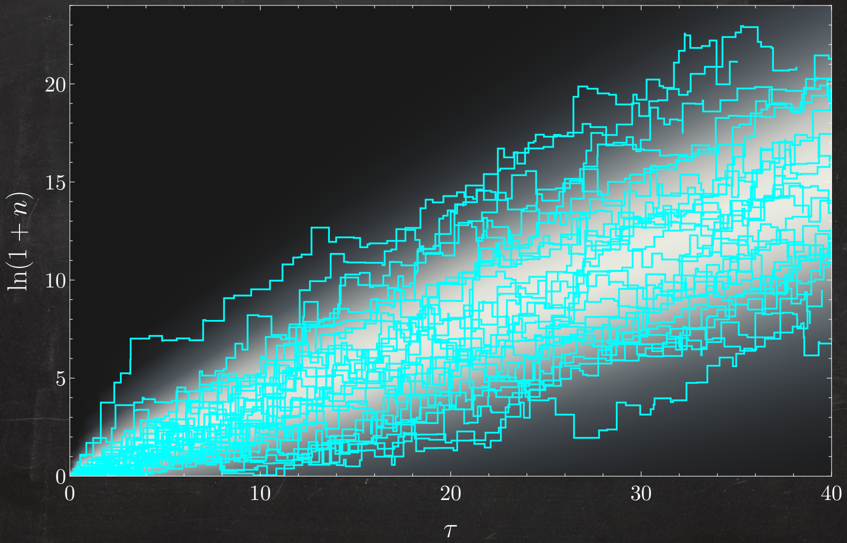
with solution

$$P(n;\tau)\, dn = \frac{1}{\sqrt{4\pi\sigma^2\tau}} \exp\left[-\frac{(\ln n - \sigma^2\tau)^2}{4\sigma^2\tau} \right] d\ln n$$

$$\Rightarrow \quad n = e^{\sigma^2\tau} - 1$$



20
15
10
5
0
0
10
20
30
40
$\ln(1+n)$
τ

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi+\psi)} \end{bmatrix}$$

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi)=e^{-\frac{i}{2}\phi}\begin{bmatrix}\cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)}\\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)}\end{bmatrix}$$

$\Rightarrow$ need 27 correlators

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix}\cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)}\\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)}\end{bmatrix}$$

$\Rightarrow$ need 27 correlators

Let

$$\langle(\Lambda_{11})^2\rangle_{\delta\tau} = \sigma_1^2\,, \qquad \langle(\Lambda_{22})^2\rangle_{\delta\tau} = \sigma_2^2\,, \qquad \langle(\Lambda_{12})^2\rangle_{\delta\tau} = \langle(\Lambda_{12})^2\rangle_{\delta\tau} = \sigma_\perp^2\,.$$

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi+\psi)} \end{bmatrix}$$

$\Rightarrow$ need 27 correlators

$$\langle \delta f_1^{(1)}\delta f_1^{(1)}\rangle = \tilde f_1^2\,\gamma_1(\theta),$$
$$\langle \delta f_1^{(1)}\delta f_2^{(1)}\rangle = 2\tilde f_1\tilde f_2\cos(2\psi)\,\gamma_3(\theta),$$
$$\langle \delta f_1^{(1)}\delta\theta^{(1)}\rangle = -\frac{2\tilde f_1}{\Delta f}\left[\tilde f_1+\tilde f_2\cos(2\psi)\right]\gamma_4(\theta),$$
$$\langle \delta f_1^{(1)}\delta\psi^{(1)}\rangle = -\tilde f_1\sin(2\psi)\left(\frac{f_2}{\tilde f_2}\gamma_3(\theta)-2\frac{\tilde f_2}{\Delta f}\gamma_4(\theta)\cot\theta\right),$$
$$\langle \delta f_1^{(1)}\delta\varphi^{(1)}\rangle = -2\frac{\tilde f_1\tilde f_2}{\Delta f}\sin(2\psi)\gamma_4(\theta)\csc\theta,$$
$$\langle \delta f_1^{(1)}\delta\phi^{(1)}\rangle = \tilde f_1\frac{f_2}{\tilde f_2}\sin(2\psi)\gamma_3(\theta),$$
$$\langle \delta f_2^{(1)}\delta f_2^{(1)}\rangle = \tilde f_2^2\,\gamma_2(\theta),$$
$$\langle \delta f_2^{(1)}\delta\theta^{(1)}\rangle = -\frac{2\tilde f_2}{\Delta f}\left[\tilde f_2+\tilde f_1\cos(2\psi)\right]\gamma_5(\theta),$$
$$\langle \delta f_2^{(1)}\delta\psi^{(1)}\rangle = -\tilde f_2\sin(2\psi)\left(\frac{f_1}{\tilde f_1}\gamma_3(\theta)-2\frac{\tilde f_1}{\Delta f}\gamma_5(\theta)\cot\right.$$ [redacted]
$$\langle \delta f_2^{(1)}\delta\varphi^{(1)}\rangle = -2\frac{\tilde f_1\tilde f_2}{\Delta f}\sin(2\psi)\gamma_5(\theta)\csc\theta,$$
$$\langle \delta f_2^{(1)}\delta\phi^{(1)}\rangle = -\tilde f_2\frac{f_1}{\tilde f_1}\sin(2\psi)\gamma_3(\theta),$$
$$\langle \delta\theta^{(1)}\delta\theta^{(1)}\rangle = 2\sigma_\perp^2+\frac{1}{\Delta f^2}\left(\tilde f_1^2+\tilde f_2^2+2\tilde f_1\tilde f_2\cos(2\psi)\right)\gamma_6(\theta),$$
$$\langle \delta\theta^{(1)}\delta\psi^{(1)}\rangle = \frac{1}{\Delta f}\left[\frac{f_1\tilde f_2}{\tilde f_1}\gamma_4(\theta)+\frac{\tilde f_1 f_2}{\tilde f_2}\gamma_5(\theta)\right]\sin(2\psi)-\cos\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle \delta\theta^{(1)}\delta\varphi^{(1)}\rangle = \frac{2\tilde f_1\tilde f_2\sin(2\psi)}{\Delta f^2}\gamma_6(\theta)\csc\theta,$$
$$\langle \delta f_1^{(2)}\rangle = \frac{1}{\Delta f}\left[\tilde f_1^2\, l_1(\theta)-(f_1f_2-1)l_3(\theta)\right],$$

$$\langle \delta\theta^{(1)}\delta\phi^{(1)}\rangle = \frac{1}{4\Delta f}\left[\left(f_1\frac{\tilde f_2}{\tilde f_1}-f_2\frac{\tilde f_1}{\tilde f_2}\right)(\sigma_1^2-\sigma_2^2)\sin\theta+4\left(f_1\frac{\tilde f_2}{\tilde f_1}+f_2\frac{\tilde f_1}{\tilde f_2}\right.\right.$$ [redacted] $$\left.\right]\sin(2\psi)$$
$$\langle \delta\psi^{(1)}\delta\psi^{(1)}\rangle = \frac{1}{4}\left[\left(\frac{f_1}{\tilde f_1}\right)^2\gamma_1(\theta)+\left(\frac{f_2}{\tilde f_2}\right)^2\gamma_2(\theta)\right]-$$ [redacted] $$\left.\sigma_2^2\right)+2\sin^2\theta\sigma_\perp^2-\cos\theta\left(2\langle\delta\psi^{(1)}\delta\varphi^{(1)}\rangle+\cos\theta\,\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle\right),$$
$$\langle \delta\psi^{(1)}\delta\varphi^{(1)}\rangle = -\frac{1}{\Delta f}\left(\frac{f_1\tilde f_2}{\tilde f_1}\gamma_4(\theta)\right.$$ [redacted] $$\left(\sigma_1^2+\sigma_2^2\right)-3\sigma_\perp^2\Big]\cos\theta-\cos\theta\,\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle \delta\psi^{(1)}\delta\phi^{(1)}\rangle$$ [redacted] $$\theta-\langle\delta\varphi^{(1)}\delta\phi^{(1)}\rangle\cos\theta,$$

[redacted] $$-\sigma_2^2)\left[5(f_1-f_2)+\left(\tilde f_1\frac{f_2}{\tilde f_2}-\tilde f_2\frac{f_1}{\tilde f_1}\right)\cos(2\psi)\right],$$
[redacted] $$\frac{}{4}\left(\frac{f_1^2}{\tilde f_1^2}-\frac{f_2^2}{\tilde f_2^2}\right)\cos\theta-\frac{1}{2}\left[\frac{f_1^2}{\tilde f_1^2}+\frac{f_2^2}{\tilde f_2^2}-2\frac{f_1}{\tilde f_1}\frac{f_2}{\tilde f_2}\cos(2\psi)\right]\gamma_3(\theta),$$
[redacted] $$\Big],$$
[redacted] $$-\sigma_2^2)\sin\theta-\frac{1}{2}\sin(2\theta)\left(\sigma_1^2+\sigma_2^2-3\sigma_\perp^2\right)+\frac{1}{\Delta f}\left(\langle\delta f_2^{(1)}\delta\theta^{(1)}\rangle-\langle\delta f_1^{(1)}\delta\theta^{(1)}\rangle\right)+\frac{1}{4}\sin(2\theta)\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle,$$
[redacted] $$\frac{}{8}\left(\frac{\tilde f_1}{\tilde f_2}+\frac{\tilde f_2}{\tilde f_1}\right)\left(\sigma_1^2+\sigma_2^2-2\sigma_\perp^2\right)\sin^2\theta\sin(2\psi)-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\phi^{(1)}\rangle-\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\phi^{(1)}\rangle\right)-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\psi^{(1)}\rangle+\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\psi^{(1)}\rangle\right)$$
$$-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\varphi^{(1)}\rangle+\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\varphi^{(1)}\rangle\right)\cos\theta+\frac{1}{2}\sin\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle-\cos\theta\langle\delta\varphi^{(2)}\rangle$$
$$-\frac{1}{2}\left(\frac{\tilde f_1}{\tilde f_2}-\frac{\tilde f_2}{\tilde f_1}\right)\left[\frac{1}{4}\left(\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle\sin^2\theta-\langle\delta\theta^{(1)}\delta\theta^{(1)}\rangle\right)\sin 2\psi+\frac{1}{2}\langle\delta\varphi^{(1)}\delta\theta^{(1)}\rangle\sin\theta\cos 2\psi\right],$$
$$\langle\delta\varphi^{(2)}\rangle = \frac{1}{\Delta f}\left(\langle\delta f_2^{(1)}\delta\varphi^{(1)}\rangle-\langle\delta f_1^{(1)}\delta\varphi^{(1)}\rangle\right)-\cot\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle\delta\phi^{(2)}\rangle = 0$$

CENSORED

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix}\cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)}\end{bmatrix}$$

Solution of full FP equation can be bypassed,

$$\partial_\tau\langle\ln(1+n)\rangle = \left\langle \frac{1}{2(1+n)}\sum_{a=1}^{N_{\rm f}}\frac{\langle\delta f_a\rangle_{\delta\tau}}{\delta\tau} - \frac{1}{8(1+n)^2}\sum_{a,b=1}^{N_{\rm f}}\frac{\langle\delta f_a\delta f_b\rangle_{\delta\tau}}{\delta\tau}\right\rangle$$
$$\xrightarrow{\tau\to\infty}\left\langle l(\theta)-\frac{1}{2}\gamma(\theta)\right\rangle$$

with

$$\gamma(\theta) = 2\left[\cos^4\left(\tfrac{\theta}{2}\right)\sigma_1^2+\sin^4\left(\tfrac{\theta}{2}\right)\sigma_2^2+4\sin^2\left(\tfrac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\sigma_\perp^2\right]$$
$$l(\theta) = 2\left[\cos^2\left(\tfrac{\theta}{2}\right)\sigma_1^2+\sin^2\left(\tfrac{\theta}{2}\right)\sigma_2^2+\sigma_\perp^2\right]$$

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix}\cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)}\\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)}\end{bmatrix}$$

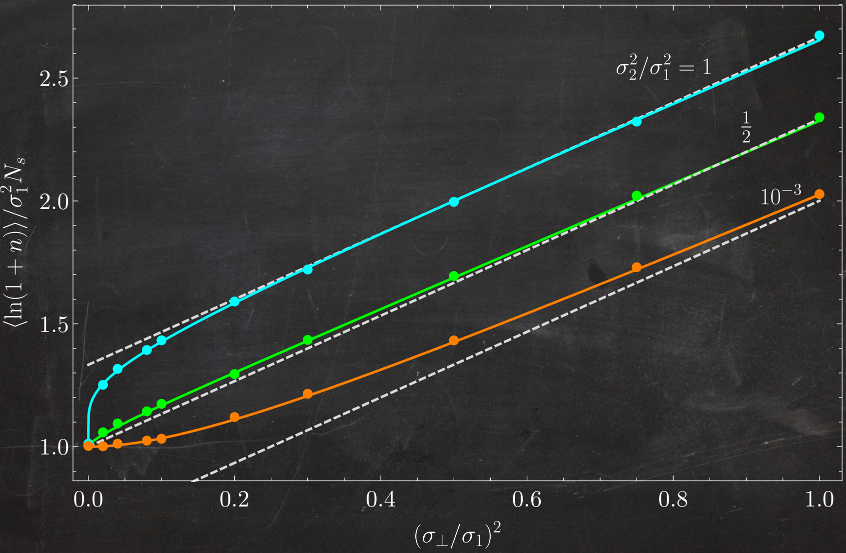
Solution of full FP equation can be bypassed,

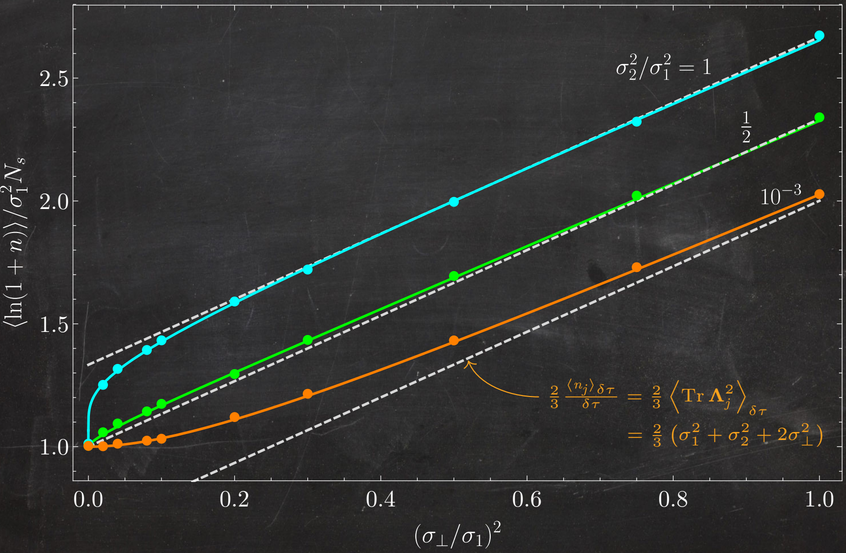
$$\begin{aligned}\partial_\tau\langle\ln(1+n)\rangle &= \left\langle \frac{1}{2(1+n)}\sum_{a=1}^{N_\mathrm{f}}\frac{\langle\delta f_a\rangle_{\delta\tau}}{\delta\tau} - \frac{1}{8(1+n)^2}\sum_{a,b=1}^{N_\mathrm{f}}\frac{\langle\delta f_a\delta f_b\rangle_{\delta\tau}}{\delta\tau}\right\rangle\\ &\xrightarrow{\tau\to\infty}\left\langle l(\theta)-\frac{1}{2}\gamma(\theta)\right\rangle\end{aligned}$$

with

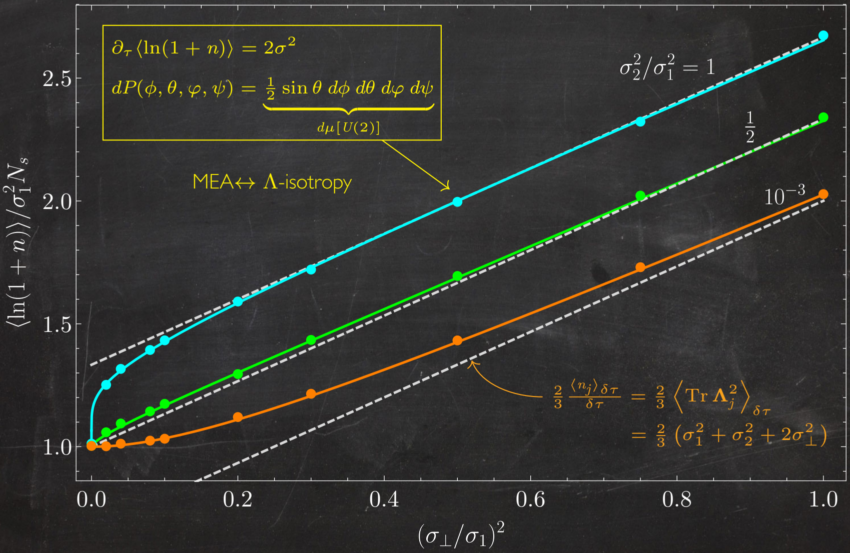
$$\int P\,d\mathbf{f}\,d\phi\,d\varphi\,d\psi = \frac{1}{\mathcal{N}}\,\frac{\sin\theta}{Q\sin^2\theta+1}\,\exp\left[2\nu\,\mathrm{arctanh}\left(\sqrt{\frac{Q}{1+Q}}\,\cos\theta\right)\right]$$

$$Q = \frac{\sigma_1^2+\sigma_2^2-4\sigma_\perp^2}{8\sigma_\perp^2},\qquad \nu = \frac{|\sigma_1^2-\sigma_2^2|}{8\sigma_\perp^2\sqrt{|Q|(1+Q)}}$$


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s$
$(\sigma_\perp/\sigma_1)^2$
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0



$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\frac{2}{3}\frac{\langle n_j\rangle_{\delta\tau}}{\delta\tau} = \frac{2}{3}\left\langle \mathrm{Tr}\,\mathbf{\Lambda}_j^2\right\rangle_{\delta\tau}$
$= \frac{2}{3}\left(\sigma_1^2+\sigma_2^2+2\sigma_\perp^2\right)$
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
$\langle \ln(1+n)\rangle/\sigma_1^2 N_s$
$(\sigma_\perp/\sigma_1)^2$



$\partial_\tau \langle \ln(1+n) \rangle = 2\sigma^2$
$dP(\phi,\theta,\varphi,\psi) = \underbrace{\frac{1}{2}\sin\theta\, d\phi\, d\theta\, d\varphi\, d\psi}_{d\mu[U(2)]}$
MEA$\leftrightarrow$ Λ-isotropy
$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\frac{2}{3}\frac{\langle n_j\rangle_{\delta\tau}}{\delta\tau} = \frac{2}{3}\left\langle \mathrm{Tr}\,\boldsymbol{\Lambda}_j^2 \right\rangle_{\delta\tau}$
$= \frac{2}{3}\left(\sigma_1^2 + \sigma_2^2 + 2\sigma_\perp^2\right)$
$\langle \ln(1+n)\rangle/\sigma_1^2 N_s$
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
$(\sigma_\perp/\sigma_1)^2$

N_f fields

$N_\mathrm{f}(N_\mathrm{f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_\mathrm{f}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2\leqslant k\leqslant N} \mathbf{A}(k)\right)\cdot[SU(N-1)]\cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}}\,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\,e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

N_f fields

$N_\mathrm{f}(N_\mathrm{f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_\mathrm{f}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2 \leqslant k \leqslant N} \mathbf{A}(k) \right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}} \,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3 \alpha_{(2k-3)}} \, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

$\Rightarrow$ need $\mathcal{O}({N_\mathrm{f}}^4)$ correlators!

N_f fields

$N_\mathrm{f}(N_\mathrm{f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_\mathrm{f}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2\leqslant k\leqslant N} \mathbf{A}(k)\right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}}\,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

$\Rightarrow$ need $\mathcal{O}({N_\mathrm{f}}^4)$ correlators! Assume

$$\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_1^2/2 & \sigma_\perp^2 & \cdots & \sigma_\perp^2 \\ \sigma_\perp^2 & \sigma_2^2/2 & \cdots & \sigma_\perp^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\perp^2 & \sigma_\perp^2 & \cdots & \sigma_2^2/2 \end{pmatrix}$$

$\Rightarrow$ Results depend only on the angle $\theta = \alpha_{2(N_\mathrm{f}-1)}$ and

$$\mathcal{F}(\Omega_{N_\mathrm{f}}) = \mathcal{F}(\Omega_{N_\mathrm{f}-1})\cos^4(\alpha_{2(N_\mathrm{f}-2)}/2) + \sin^4(\alpha_{2(N_\mathrm{f}-2)}/2)$$

with $\mathcal{F}(\Omega_2) = 1$

N_{f} fields

$N_{\mathrm{f}}(N_{\mathrm{f}}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_{\mathrm{f}}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2\leqslant k\leqslant N} \mathbf{A}(k)\right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}}\,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

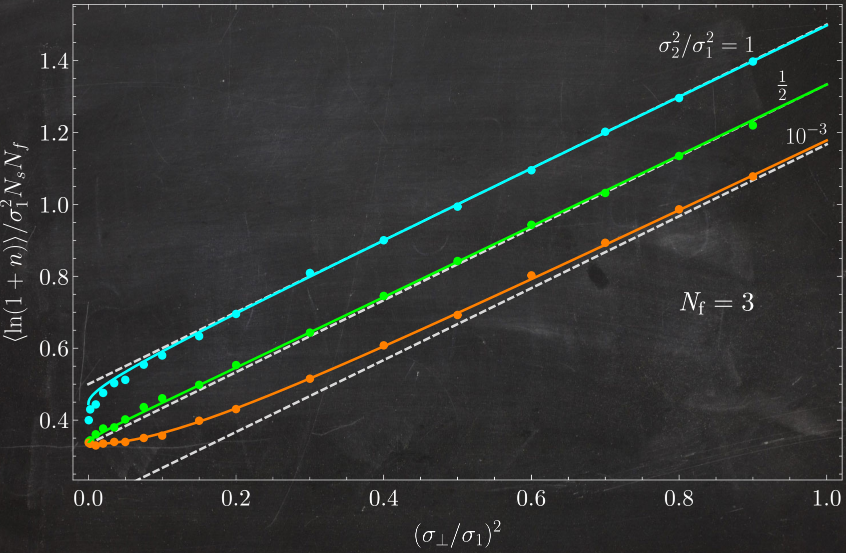
$\Rightarrow$ need $\mathcal{O}({N_{\mathrm{f}}}^4)$ correlators! Assume

$$\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_1^2/2 & \sigma_\perp^2 & \cdots & \sigma_\perp^2 \\ \sigma_\perp^2 & \sigma_2^2/2 & \cdots & \sigma_\perp^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\perp^2 & \sigma_\perp^2 & \cdots & \sigma_2^2/2 \end{pmatrix}$$

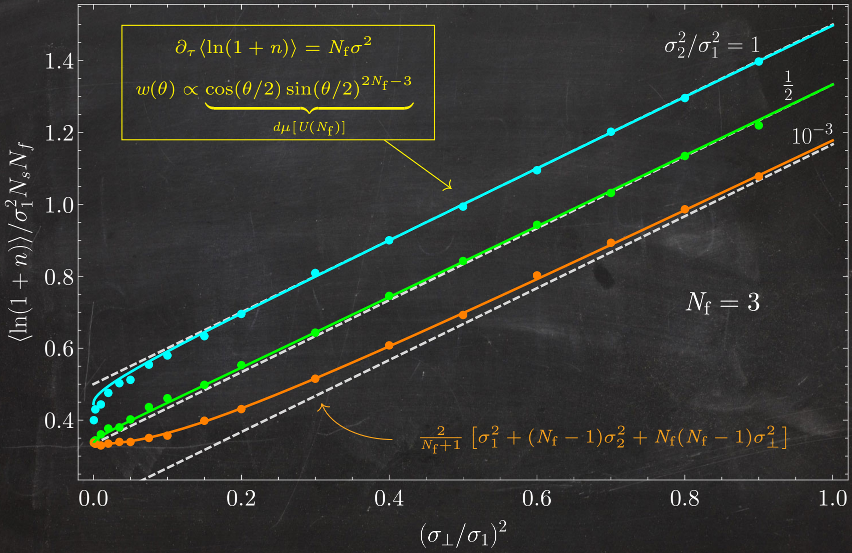
$$\partial_\tau \langle \ln(1+n)\rangle = \left\langle l(\theta) - \frac{1}{2}\gamma(\theta,\Omega_{N_{\mathrm{f}}})\right\rangle$$

$$l(\theta) = 2\left[\sigma_1^2\cos^2(\theta/2) + \sigma_2^2\sin^2(\theta/2) + \sigma_\perp^2(N_{\mathrm{f}}-1)\right]$$

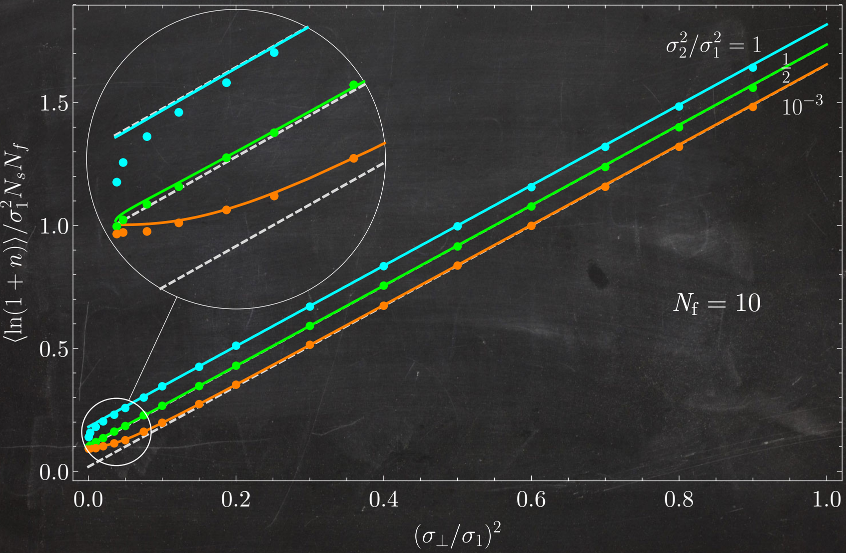
$$\gamma(\theta,\Omega_{N_{\mathrm{f}}}) = 2\Big[\sigma_1^2\cos^4(\theta/2) + \sigma_2^2\sin^4(\theta/2)\,\mathcal{F}_\Omega + 2\sigma_\perp^2\left(1-\cos^4(\theta/2) - \sin^4(\theta/2)\,\mathcal{F}_\Omega\right)\Big]$$

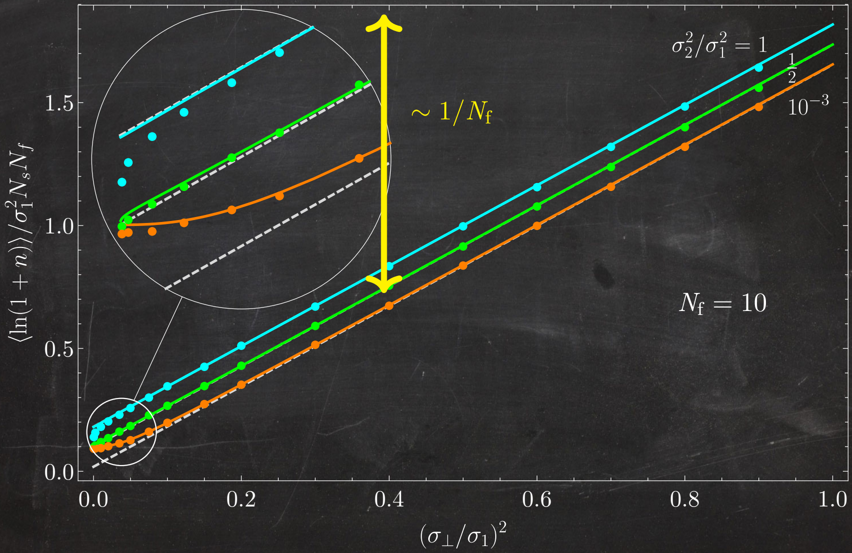


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_\mathrm{f} = 3$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
1.4
1.2
1.0
0.8
0.6
0.4
0.0
0.2
0.4
0.6
0.8
1.0



$\partial_\tau \langle \ln(1+n) \rangle = N_{\mathrm{f}} \sigma^2$
$w(\theta) \propto \underbrace{\cos(\theta/2)\sin(\theta/2)^{2N_{\mathrm{f}}-3}}_{d\mu[U(N_{\mathrm{f}})]}$
$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_{\mathrm{f}} = 3$
$\frac{2}{N_{\mathrm{f}}+1}\left[\sigma_1^2 + (N_{\mathrm{f}}-1)\sigma_2^2 + N_{\mathrm{f}}(N_{\mathrm{f}}-1)\sigma_\perp^2\right]$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
1.4
1.2
1.0
0.8
0.6
0.4
0.0
0.2
0.4
0.6
0.8
1.0


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_{\rm f} = 10$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0


$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\sim 1/N_\mathrm{f}$
$N_\mathrm{f} = 10$
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0

Conclusion

- Avoid relying on detailed model building, and take a coarse grained approach to the particle production in the early universe
- MEA captures the universal features arising from a Central Limit Threorem (concentration of measure)…
- …as long as there's no hierarchy of couplings
- Break from weak scattering limit $\Rightarrow$ Random Matrix Theory?
- Next: include expansion and metric perturbations

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Thank you