The H_3 integrable system

Marcos A. G. García

(Collaboration with A. Turbiner)

Instituto de Ciencias Nucleares, UNAM, México

August 10, 2010

Rational integrable systems

- The Hamiltonian Reduction method provides an opportunity to construct non-trivial multidimensional completely integrable quantum Hamiltonians.
- These Hamiltonians are associated to the root spaces of the classical (A_n, B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras
- In the case of rational potentials one can also construct Hamiltonians associated with the noncrystallographic systems H_3, H_4 and $I_2(m)$ (Olshanestky-Perelomov, '75)

Rational integrable systems

- The Hamiltonian Reduction method provides an opportunity to construct non-trivial multidimensional completely integrable quantum Hamiltonians.
- These Hamiltonians are associated to the root spaces of the classical (A_n, B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras
- In the case of rational potentials one can also construct Hamiltonians associated with the noncrystallographic systems H_3, H_4 and $I_2(m)$ (Olshanestky-Perelomov, '75)

Rational integrable systems

- The Hamiltonian Reduction method provides an opportunity to construct non-trivial multidimensional completely integrable quantum Hamiltonians.
- These Hamiltonians are associated to the root spaces of the classical (A_n, B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras
- In the case of rational potentials one can also construct Hamiltonians associated with the noncrystallographic systems H_3, H_4 and $I_2(m)$ (Olshanestky-Perelomov, '75)

- Algebraic expressions of quantum Hamiltonians for all crystallographic root systems have been found for both rational and trigonometric cases (A. Turbiner et al., 1997 - 2009)
- The eigenfunctions can be obtained explicitly as polynomials
- The spectrum can be found in a closed form as a polynomial in the quantum numbers

- Algebraic expressions of quantum Hamiltonians for all crystallographic root systems have been found for both rational and trigonometric cases (A. Turbiner et al., 1997 - 2009)
- The eigenfunctions can be obtained explicitly as polynomials
- The spectrum can be found in a closed form as a polynomial in the quantum numbers

- Algebraic expressions of quantum Hamiltonians for all crystallographic root systems have been found for both rational and trigonometric cases (A. Turbiner et al., 1997 - 2009)

- The eigenfunctions can be obtained explicitly as polynomials

- The spectrum can be found in a closed form as a polynomial in the quantum numbers

The Hamiltonian in the rational case is

$$\mathcal{H}_\Delta = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] + \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} g_{|\alpha|}|\alpha|^2\frac{1}{(\alpha\cdot x)^2}$$

- $\mathcal{R}_+$ = set of positive roots in the system Δ, $\mathrm{rank}(\Delta) = N$
- $\omega \in \mathbf{R}$ a parameter
- $g_{|\alpha|}$ coupling constants depending on the root length
- $x = (x_1, x_2, \ldots, x_N)$

Configuration space is the subspace of $\mathbf{R}^N$ where

$$(\alpha\cdot x) > 0$$

for any $\alpha \in \mathcal{R}_+$

The Hamiltonian in the rational case is

$$\mathcal{H}_\Delta = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] + \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} g_{|\alpha|}|\alpha|^2\frac{1}{(\alpha\cdot x)^2}$$

- $\mathcal{R}_+$ = set of positive roots in the system Δ, $\mathrm{rank}(\Delta) = N$
- $\omega \in \mathbf{R}$ a parameter
- $g_{|\alpha|}$ coupling constants depending on the root length
- $x = (x_1, x_2, \ldots, x_N)$

Configuration space is the subspace of $\mathbf{R}^N$ where

$$(\alpha\cdot x) > 0$$

for any $\alpha \in \mathcal{R}_+$

The goal is

- To find a transformation $x \to \tau$ leading to *algebraic* form of the Hamiltonian (if exists)

$$h_{\Delta} = \sum_{i,j=1}^{N} A_{ij}(\tau) \frac{\partial^2}{\partial \tau_i \partial \tau_j} + \sum_{j=1}^{N} B_j(\tau) \frac{\partial}{\partial \tau_j} \ ,$$

where $A_{ij}(\tau)$, $B_j(\tau)$ are polynomials

- Find finite-dimensional invariant spaces for h_Δ of a form

$$\mathcal{P}_n{}^{\boldsymbol{\alpha}} = \langle \tau_1^{p_1}\tau_2^{p_2}\dots\tau_r^{p_r} | 0 \le \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n \rangle \ ,$$

(if exist) for

$$n = 0, 1, 2, \dots \quad \alpha_1, \alpha_2, \dots, \alpha_r \in \mathbf{Z}^+$$

They are classified by *characteristic vector*

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r)$$

- These spaces can be ordered by inclusion:

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$$

Such an object is called an *infinite flag*

- Find finite-dimensional invariant spaces for h_Δ of a form

$$\mathcal{P}_n{}^{\boldsymbol{\alpha}} = \langle \tau_1^{p_1}\tau_2^{p_2}\dots\tau_r^{p_r} | 0 \leq \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_r p_r \leq n \rangle \ ,$$

(if exist) for

$$n = 0, 1, 2, \dots \quad \alpha_1, \alpha_2, \dots, \alpha_r \in \mathbf{Z}^+$$

They are classified by *characteristic vector*

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r)$$

- These spaces can be ordered by inclusion:

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots \subset \mathcal{P}_n \subset \dots$$

Such an object is called an *infinite flag*

The H_3 rational model

The H_3 rational Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{g}{x_k^2}\right] + \sum_{\{i,j,k\}}\sum_{\mu_{1,2}=0,1}\frac{2g}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]^2}$$

where $\{i,j,k\} = \{1,2,3\}$ and all even permutations. The coupling constant is

$$g = \nu(\nu - 1) > -\frac{1}{4}$$

and

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Explicitly:

$$\mathcal{H} = -\frac{1}{2}\Delta^{(3)} + \frac{1}{2}\omega^2(x_1^2+x_2^2+x_3^2) + \frac{1}{2}\nu(\nu-1)\left[\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}\right]$$
$$+ 2\nu(\nu-1)\left[\frac{1}{(x_1+\varphi_+x_2+\varphi_-x_3)^2} + \frac{1}{(x_1-\varphi_+x_2+\varphi_-x_3)^2}\right.$$
$$+ \frac{1}{(x_1+\varphi_+x_2-\varphi_-x_3)^2} + \frac{1}{(x_1-\varphi_+x_2-\varphi_-x_3)^2} + \frac{1}{(x_2+\varphi_+x_3+\varphi_-x_1)^2}$$
$$+ \frac{1}{(x_2-\varphi_+x_3+\varphi_-x_1)^2} + \frac{1}{(x_2+\varphi_+x_3-\varphi_-x_1)^2} + \frac{1}{(x_2-\varphi_+x_3-\varphi_-x_1)^2}$$
$$+ \frac{1}{(x_3+\varphi_+x_1+\varphi_-x_2)^2} + \frac{1}{(x_3-\varphi_+x_1+\varphi_-x_2)^2} + \frac{1}{(x_3+\varphi_+x_1-\varphi_-x_2)^2}$$
$$\left.+\frac{1}{(x_3-\varphi_+x_1-\varphi_-x_2)^2}\right]$$

The Hamiltonian is invariant wrt the H_3 Coxeter group, which is the full symmetry group of the icosahedron.

The Hamiltonian is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j$$

$$\varphi_+ \longleftrightarrow \varphi_-$$

The ground state function and its eigenvalue are

$$\Psi_0 = \Delta_1^{\nu}\Delta_2^{\nu}\exp\left(-\frac{\omega}{2}\sum_{k=1}^{3}x_k^2\right)\ , \quad E_0 = \frac{3}{2}\omega(1+10\nu)$$

where

$$\Delta_1 = \prod_{k=1}^{3} x_k$$

$$\Delta_2 = \prod_{\{i,j,k\}}\prod_{\mu_{1,2}=0,1}\left[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k\right]$$

Explicitly:

$$\Psi_0 = [x_1\ x_2\ x_3]^\nu \times$$

$$[(x_1 + \varphi_+ x_2 + \varphi_- x_3)\,(x_1 - \varphi_+ x_2 + \varphi_- x_3)\,(x_1 + \varphi_+ x_2 - \varphi_- x_3)$$

$$(x_1 - \varphi_+ x_2 - \varphi_- x_3)\,(x_2 + \varphi_+ x_3 + \varphi_- x_1)\,(x_2 - \varphi_+ x_3 + \varphi_- x_1)$$

$$(x_2 + \varphi_+ x_3 - \varphi_- x_1)\,(x_2 - \varphi_+ x_3 - \varphi_- x_1)\,(x_3 + \varphi_+ x_1 + \varphi_- x_2)$$

$$(x_3 - \varphi_+ x_1 + \varphi_- x_2)\,(x_3 + \varphi_+ x_1 - \varphi_- x_2)\,(x_3 - \varphi_+ x_1 - \varphi_- x_2)\,]^\nu$$

$$\times\ \exp\left[-\frac{\omega}{2}(x_1^2 + x_2^2 + x_3^2)\right]$$

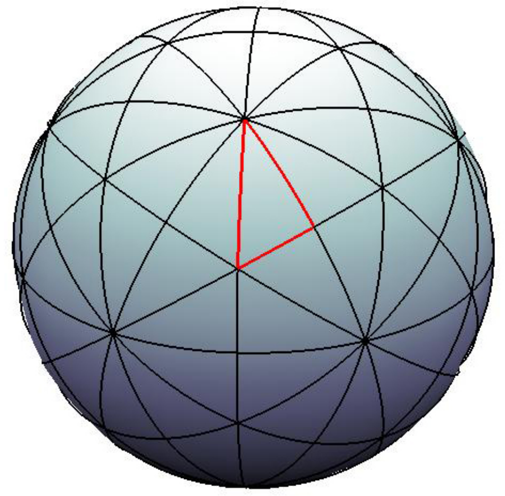
Configuration space

The configuration space is the domain in $\mathbf{R}^3$ where $x_{1,2,3} > 0$ bounded by the planes

$$x_1 = 0\ , \quad x_3 = 0\ ,$$

$$x_3 + \varphi_+ x_1 + \varphi_- x_2 = 0\ .$$

(the domain where $(\alpha \cdot x) > 0$).



The algebraic form of the Hamiltonian

Make a gauge rotation of the Hamiltonian:

$$h = -2(\Psi_0)^{-1}(\mathcal{H} - E_0)(\Psi_0)$$

New spectral problem arises

$$h\phi(x) = -2\epsilon\phi(x)$$

with spectral parameter $\epsilon = E - E_0$

Can we find variables leading to an algebraic form of h?

The algebraic form of the Hamiltonian

Make a gauge rotation of the Hamiltonian:

$$h = -2(\Psi_0)^{-1}(\mathcal{H} - E_0)(\Psi_0)$$

New spectral problem arises

$$h\phi(x) = -2\epsilon\phi(x)$$

with spectral parameter $\epsilon = E - E_0$

Can we find variables leading to an algebraic form of h?

What might those variables be?

The invariants of the H_3 group

- Consider the fundamental weights of Δ_{H_3} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, \varphi_+, 1)$	12
$\omega_2 = (1, \varphi_+^2, 0)$	20
$\omega_3 = (0, 2\varphi_+, 0)$	30

- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega_1} (\omega \cdot x)^a$$

$a = 2, 6, 10$ are the degrees of the H_3 group

What might those variables be?

The invariants of the H_3 group

- Consider the fundamental weights of Δ_{H_3} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, \varphi_+, 1)$	12
$\omega_2 = (1, \varphi_+^2, 0)$	20
$\omega_3 = (0, 2\varphi_+, 0)$	30

- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega_1} (\omega \cdot x)^a$$

$a = 2, 6, 10$ are the degrees of the H_3 group

What might those variables be?

The invariants of the H_3 group

- Consider the fundamental weights of Δ_{H_3} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, \varphi_+, 1)$	12
$\omega_2 = (1, \varphi_+^2, 0)$	20
$\omega_3 = (0, 2\varphi_+, 0)$	30

- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega_1} (\omega \cdot x)^a$$

$a = 2, 6, 10$ are the degrees of the H_3 group

What might those variables be?

The invariants of the H_3 group

- Consider the fundamental weights of Δ_{H_3} and their orbits Ω:

weight vector	orbit size
$\omega_1 = (0, \varphi_+, 1)$	12
$\omega_2 = (1, \varphi_+^2, 0)$	20
$\omega_3 = (0, 2\varphi_+, 0)$	30

- Choose the shortest orbit and average

$$t_a(x) = \sum_{\omega \in \Omega_1} (\omega \cdot x)^a$$

$a = 2, 6, 10$ are the degrees of the H_3 group

- The invariants are defined ambiguously

$$
\begin{aligned}
t_2 &\longrightarrow t_2 \\
t_6 &\longrightarrow t_6 + \alpha_1 t_2^3 \\
t_{10} &\longrightarrow t_{10} + \alpha_2 t_2^2 t_6 + \alpha_3 t_2^5
\end{aligned}
$$

- We look for parameters α_i such that
  - the Hamiltonian h has algebraic form
  - has infinitely-many invariant subspaces in polynomials
  - these subspaces form a flag
  - the flag is "minimal"

- The invariants are defined ambiguously

$$
\begin{aligned}
t_2 &\longrightarrow t_2 \\
t_6 &\longrightarrow t_6 + \alpha_1 t_2^3 \\
t_{10} &\longrightarrow t_{10} + \alpha_2 t_2^2 t_6 + \alpha_3 t_2^5
\end{aligned}
$$

- We look for parameters α_i such that
  - the Hamiltonian h has algebraic form
  - has infinitely-many invariant subspaces in polynomials
  - these subspaces form a flag
  - the flag is "minimal"

Those variables are

$$\tau_1 = \frac{1}{10+2\sqrt{5}}\ t_2$$

$$\tau_2 = \frac{1}{10+16\sqrt{5}}\left(t_6 - \frac{13}{10}t_2^3\right)$$

$$\tau_3 = \frac{1}{250+110\sqrt{5}}\left(t_{10} - \frac{76}{15}t_2^2 t_6 + \frac{1531}{375}t_2^5\right)$$

Explicit expressions

The Hamiltonian takes the algebraic form

$$h = \sum_{i,j=1}^{3} A_{ij}\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{j=1}^{3} B_j\frac{\partial}{\partial\tau_j}$$

with

$$
\begin{aligned}
A_{11} &= 4\tau_1 \\
A_{12} &= 12\tau_2 \\
A_{13} &= 20\tau_3 \\
A_{22} &= -\frac{48}{5}\tau_1^2\tau_2 + \frac{45}{2}\tau_3 \\
A_{23} &= \frac{16}{15}\tau_1\tau_2^2 - 24\tau_1^2\tau_3 \\
A_{33} &= -\frac{64}{3}\tau_1\tau_2\tau_3 + \frac{128}{45}\tau_2^3 \\
B_1 &= 6 + 60\nu - 4\omega\tau_1 \\
B_2 &= -\frac{48}{5}(1+5\nu)\tau_1^2 - 12\omega\tau_2 \\
B_3 &= -\frac{64}{15}(2+5\nu)\tau_1\tau_2 - 20\omega\tau_3
\end{aligned}
$$

Configuration space and Jacobian

In τ's the configuration space boundary is an algebraic surface of degree 7 (degree 30 in x)

$$\kappa(\tau) = -12960\tau_1^5\tau_3^2 + 5760\tau_1^4\tau_2^2\tau_3 - 640\tau_1^3\tau_2^4 - 54000\tau_1^2\tau_2\tau_3^2 + 21600\tau_1\tau_2^3\tau_3 - 2304\tau_2^5 - 50625\tau_3^3 = 0$$

Boundary corresponds to zeros of Ψ_0

The square of the Jacobian of the change of variables $x \to \tau$ can be calculated explicitly:

$$J^2 = \begin{vmatrix} \frac{\partial \tau_1}{\partial x_1} & \frac{\partial \tau_1}{\partial x_2} & \frac{\partial \tau_1}{\partial x_3} \\ \frac{\partial \tau_2}{\partial x_1} & \frac{\partial \tau_2}{\partial x_2} & \frac{\partial \tau_2}{\partial x_3} \\ \frac{\partial \tau_3}{\partial x_1} & \frac{\partial \tau_3}{\partial x_2} & \frac{\partial \tau_3}{\partial x_3} \end{vmatrix}^2 = \frac{9}{5} \prod_{\alpha \in \mathcal{R}_3^+} (\alpha \cdot x)^2 = \frac{8}{45}\, \kappa(\tau)\ .$$

It vanishes on the boundary of the configuration space.

Polynomial spaces

The algebraic operator h preserves subspaces

$$\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3} | 0 \leq n_1 + 2n_2 + 3n_3 \leq n \rangle \ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,2,3), they form flag

The flag is invariant with respect to weighted-projective transformations:

$$\tau_1 \longrightarrow \tau_1 + a$$
$$\tau_2 \longrightarrow \tau_2 + b_1\tau_1^2 + b_2\tau_1 + b_3$$
$$\tau_3 \longrightarrow \tau_3 + c_1\tau_1\tau_2 + c_2\tau_1^3 + c_3\tau_2 + c_4\tau_1^2 + c_5\tau_1 + c_6$$

Polynomial spaces

The algebraic operator h preserves subspaces

$$\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3} | 0 \leq n_1 + 2n_2 + 3n_3 \leq n \rangle \ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,2,3), they form flag

The flag is invariant with respect to weighted-projective transformations:

$$\begin{aligned}
\tau_1 &\longrightarrow \tau_1 + a \\
\tau_2 &\longrightarrow \tau_2 + b_1\tau_1^2 + b_2\tau_1 + b_3 \\
\tau_3 &\longrightarrow \tau_3 + c_1\tau_1\tau_2 + c_2\tau_1^3 + c_3\tau_2 + c_4\tau_1^2 + c_5\tau_1 + c_6
\end{aligned}$$

Eigenfunctions and spectrum

One can find the spectrum of h explicitly:

$$\epsilon_{k_1,k_2,k_3} = 2\omega(k_1 + 3k_2 + 5k_3)\ , \quad k_i = 0, 1, 2, \ldots$$

Degeneracy: $\quad k_1 + 3k_2 + 5k_3 = \text{integer}$

The energies of the original Hamiltonian are

$$E = E_0 + \epsilon$$

Eigenfunctions $\phi_{n,i}$ of h are elements of $\mathcal{P}_n^{(1,2,3)}$.
Eigenfunctions of $\mathcal{H}$ are

$$\Psi = \Psi_0\ \phi \qquad \text{(factorization)}$$

- $\mathbf{n=0}$:

$$\phi_{0,0} = 1\ , \qquad \epsilon_{0,0} = 0\ .$$

- $\mathbf{n=1}$:

$$\phi_{1,0} = \tau_1 + \frac{3}{2\omega}(1+10\nu)\ , \qquad \epsilon_{1,0} = 2\omega\ .$$

- $\mathbf{n=2}$:

$$\begin{aligned} \phi_{2,0} &= \tau_1^2 - \frac{5}{\omega}(1+6\nu)\tau_1 + \frac{15}{4\omega^2}(1+6\nu)(1+10\nu)\,, \\ \epsilon_{2,0} &= 4\omega\,, \end{aligned}$$

$$\begin{aligned} \phi_{2,1} &= \tau_2 + \frac{12}{5\omega}(1+5\nu)\tau_1^2 - \frac{6}{\omega^2}(1+5\nu)(1+6\nu)\tau_1 \\ &\quad + \frac{3}{\omega^3}(1+5\nu)(1+6\nu)(1+10\nu)\,, \\ \epsilon_{2,1} &= 6\omega\,. \end{aligned}$$

Separation in spherical coordinates

In spherical coordinates the H_3 Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\Delta^{(3)} + \frac{1}{2}\omega^2 r^2 + \frac{W(\theta,\phi)}{r^2}$$

The Schroedinger equation is separable:

$$\Psi(r,\theta,\phi) = R(r)Q(\theta,\phi)$$

with

$$\left[-\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{2}\omega^2 r^2 + \frac{\gamma}{r^2}\right] R(r) = ER(r)$$

$$\left[\frac{1}{2}\,\mathcal{L}^2 + W(\theta,\phi)\right]\;Q(\theta,\phi) = \gamma\;Q(\theta,\phi)$$

The operator

$$\mathcal{F} = \frac{1}{2}\,\mathcal{L}^2 + W(\theta,\phi)$$

▸ Explicit expression

is an integral of motion

$$[\mathcal{H},\mathcal{F}] = 0$$

The gauge rotated operator

$$f = (\Psi_0)^{-1}(\mathcal{F}-\gamma_0)\Psi_0\ , \quad \gamma_0 = \frac{15}{2}\nu(1+15\nu)$$

has an algebraic form in τ's:

$$f = \sum_{i,j=2}^{3} F_{ij}\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{j=2}^{3} G_j\frac{\partial}{\partial\tau_j}$$

with

$$
\begin{aligned}
F_{22} =& \frac{24}{5}\tau_1^3\tau_2 - \frac{45}{4}\tau_1\tau_3 + 18\tau_2^2 \ , \\
F_{23} =& -\frac{8}{15}\tau_1^2\tau_2^2 + 12\tau_1^3\tau_3 + 30\tau_2\tau_3 \ , \\
F_{33} =& -\frac{64}{45}\tau_1\tau_2^3 + \frac{32}{3}\tau_1^2\tau_2\tau_3 + 50\tau_3^2 \ , \\
G_2 =& \frac{24}{5}(1+5\nu)\tau_1^3 + 3(7+30\nu)\tau_2 \ , \\
G_3 =& \frac{32}{15}(2+5\nu)\tau_1^2\tau_2 + 5(11+30\nu)\tau_3 \ .
\end{aligned}
$$

The integral f preserves the flag of polynomials

$$\mathcal{P}_n^{(1,3,5)} = \langle \tau_1^{n_1} \tau_2^{n_2} \tau_3^{n_3} | 0 \leq n_1 + 3n_2 + 5n_3 \leq n \rangle$$

hence characteristic vector is (1,3,5).The operator h also preserves this flag.

Joint eigenfunctions belong to this flag !

Existence of the flag (1,2,3) may be related to degeneracy of $\mathcal{H}$

The $h^{(3)}$ algebra.

Can $\mathcal{P}_n^{(1,2,3)}$ be finite-dimensional representation spaces of a Lie algebra of differential operators? Yes

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated (30 operators).

Generating elements can be split in two classes:

- First class: *lowering and Cartan operators*, they act on $\mathcal{P}_n$ at any n, infinite flag is preserved
- Second class: *raising operators*, a single space at a certain n is preserved

The $h^{(3)}$ algebra.

Can $\mathcal{P}_n^{(1,2,3)}$ be finite-dimensional representation spaces of a Lie algebra of differential operators? Yes

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated (30 operators).

Generating elements can be split in two classes:

- First class: *lowering and Cartan operators*, they act on $\mathcal{P}_n$ at any n, infinite flag is preserved
- Second class: *raising operators*, a single space at a certain n is preserved

The $h^{(3)}$ algebra.

Can $\mathcal{P}_n^{(1,2,3)}$ be finite-dimensional representation spaces of a Lie algebra of differential operators? Yes

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated (30 operators).

Generating elements can be split in two classes:

- First class: *lowering and Cartan operators*, they act on $\mathcal{P}_n$ at any n, infinite flag is preserved
- Second class: *raising operators*, a single space at a certain n is preserved

The $h^{(3)}$ algebra.

Can $\mathcal{P}_n^{(1,2,3)}$ be finite-dimensional representation spaces of a Lie algebra of differential operators? Yes

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated (30 operators).

Generating elements can be split in two classes:

- First class: *lowering and Cartan operators*, they act on $\mathcal{P}_n$ at any n, infinite flag is preserved
- Second class: *raising operators*, a single space at a certain n is preserved

First order operators

The first class generators consist of 13 first order operators

$$
\begin{array}{lll}
T_0^{(1)} = \partial_1 \,, & T_0^{(2)} = \partial_2 \,, & T_0^{(3)} = \partial_3 \,, \\
T_1^{(1)} = \tau_1 \partial_1 \,, & T_2^{(2)} = \tau_2 \partial_2 \,, & T_3^{(3)} = \tau_3 \partial_3 \,, \\
T_1^{(3)} = \tau_1 \partial_3 \,, & T_{11}^{(3)} = \tau_1^2 \partial_3 \,, & T_{111}^{(3)} = \tau_1^3 \partial_3 \,, \\
T_1^{(2)} = \tau_1 \partial_2 \,, & T_{11}^{(2)} = \tau_1^2 \partial_2 \,, & T_2^{(3)} = \tau_2 \partial_3 \,, \\
T_{12}^{(3)} = \tau_1 \tau_2 \partial_3 & &
\end{array}
$$

Second and third order operators

plus 6 second order generators

$$T_2^{(11)} = \tau_2 \partial_{11}\,, \qquad T_{22}^{(13)} = \tau_2^2 \partial_{13}\,, \qquad T_{222}^{(33)} = \tau_2^3 \partial_{33}\,,$$
$$T_3^{(12)} = \tau_3 \partial_{12}\,, \qquad T_3^{(22)} = \tau_3 \partial_{22}\,, \qquad T_{13}^{(22)} = \tau_1 \tau_3 \partial_{22}$$

and 2 third order generators

$$T_3^{(111)} = \tau_3 \partial_{111}\,, \qquad T_{33}^{(222)} = \tau_3^2 \partial_{222}$$

These 21 operators are generating elements of the flag-preserving subalgebra of $h^{(3)}$

Second class (raising operators)

Define the auxiliary operator (which belongs to the first class)

$$J_0 = \tau_1\partial_1 + 2\tau_2\partial_2 + 3\tau_3\partial_3 - n$$

Raising generators consist of 8 operators of 1st, 2nd and 3rd order

$$J_1^+ = \tau_1 J_0\,, \qquad J_{2,-1}^+ = \tau_2\partial_1 J_0\,, \qquad J_{3,-2}^+ = \tau_3\partial_2 J_0\,,$$
$$J_2^+ = \tau_2 J_0(J_0+1)\,, \qquad J_{3,-11}^+ = \tau_3\partial_{11} J_0\,, \qquad J_{22,-3}^+ = \tau_2^2\partial_3 J_0\,,$$
$$J_3^+ = \tau_3 J_0(J_0+1)(J_0+2)\,, \qquad J_{3,-1}^+ = \tau_3\partial_1 J_0(J_0+1)$$

$h^{(3)}$ is the infinite dimensional algebra of monomials in the 30 (22+8) generating elements

Subalgebras of $h^{(3)}$

Generating elements of $h^{(3)}$ can be grouped in 10 Abelian subalgebras

$$
\begin{aligned}
L = \{T_0^{(3)}, T_1^{(3)}, T_{11}^{(3)}, T_{111}^{(3)}\} &\longleftrightarrow \mathfrak{L} = \{T_3^{(111)}, J^+_{3,-11}, J^+_{3,-1}, J^+_3\} \\
R = \{T_0^{(2)}, T_1^{(2)}, T_{11}^{(2)}\} &\longleftrightarrow \mathfrak{R} = \{T_2^{(11)}, J^+_{2,-1}, J^+_2\} \\
F = \{T_2^{(3)}, T_{12}^{(3)}\} &\longleftrightarrow \mathfrak{F} = \{T_3^{(12)}, J^+_{3,-2}\} \\
E = \{T_{13}^{(22)}, T_3^{(22)}\} &\longleftrightarrow \mathfrak{E} = \{T_{22}^{(13)}, J^+_{22,-3}\} \\
G = \{T_{222}^{(33)}\} &\longleftrightarrow \mathfrak{G} = \{T_{33}^{(222)}\}
\end{aligned}
$$

and a closed subalgebra

$$B = \{T_0^{(1)}, T_1^{(1)}, T_2^{(2)}, T_3^{(3)}, J_0, J_1^+\}$$

Commutation relations between commutative algebras:

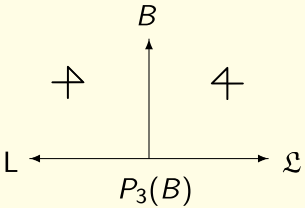
$$
\begin{aligned}
&[L, R] = 0, && [\mathfrak{L}, \mathfrak{R}] = 0,\\
&[L, F] = 0, && [\mathfrak{L}, \mathfrak{F}] = 0,\\
&[L, E] = P_2(R), && [\mathfrak{L}, \mathfrak{E}] = P_2(\mathfrak{R}),\\
&[L, G] = 0, && [\mathfrak{L}, \mathfrak{G}] = 0,\\
&[R, F] = L, && [\mathfrak{R}, \mathfrak{F}] = \mathfrak{L},\\
&[R, E] = 0, && [\mathfrak{R}, \mathfrak{E}] = 0,\\
&[R, G] = P_2(F), && [\mathfrak{R}, \mathfrak{G}] = P_2(\mathfrak{F}),\\
&[F, E] = P_2(R \oplus B), && [\mathfrak{F}, \mathfrak{E}] = P_2(\mathfrak{R} \oplus B),\\
&[F, G] = 0, && [\mathfrak{F}, \mathfrak{G}] = 0,\\
&[E, G] = P_3(F \oplus B), && [\mathfrak{E}, \mathfrak{G}] = P_3(\mathfrak{F} \oplus B),
\end{aligned}
$$

$$[L, \mathfrak{R}] = P_2(F \oplus B), \qquad [\mathfrak{L}, R] = P_2(\mathfrak{F} \oplus B),$$
$$[L, \mathfrak{F}] = P_2(R \oplus B), \qquad [\mathfrak{L}, F] = P_2(\mathfrak{R} \oplus B),$$
$$[L, \mathfrak{E}] = P_2(F), \qquad [\mathfrak{L}, E] = P_2(\mathfrak{F}),$$
$$[L, \mathfrak{G}] = P_2(R \oplus E), \qquad [\mathfrak{L}, G] = P_2(\mathfrak{R} \oplus \mathfrak{E}),$$
$$[R, \mathfrak{F}] = E, \qquad [\mathfrak{R}, F] = \mathfrak{E},$$
$$[R, \mathfrak{E}] = P_2(F \oplus B), \qquad [\mathfrak{R}, E] = P_2(\mathfrak{F} \oplus B),$$
$$[R, \mathfrak{G}] = 0, \qquad [\mathfrak{R}, G] = 0,$$
$$[F, \mathfrak{E}] = G, \qquad [\mathfrak{F}, E] = \mathfrak{G},$$
$$[F, \mathfrak{G}] = P_2(E \oplus B), \qquad [\mathfrak{F}, G] = P_2(\mathfrak{E} \oplus B),$$
$$[E, \mathfrak{G}] = 0, \qquad [\mathfrak{E}, G] = 0,$$

$$
\begin{aligned}
&[L,\mathfrak{L}] = P_3(B), \qquad && [R,\mathfrak{R}] = P_2(B), \qquad && [F,\mathfrak{F}] = P_2(B),\\
&[E,\mathfrak{E}] = P_3(B), \qquad && [G,\mathfrak{G}] = P_4(B)
\end{aligned}
$$

Commutation relations between Abelian subalgebras and B:

$$
\begin{aligned}
&[L,B] = L\,, \quad [R,B] = R\,, \quad [F,B] = F\,, \quad [E,B] = E\,, \quad [G,B] = G\,,\\
&[\mathfrak{L},B] = \mathfrak{L}\,, \quad [\mathfrak{R},B] = \mathfrak{R}\,, \quad [\mathfrak{F},B] = \mathfrak{F}\,, \quad [\mathfrak{E},B] = \mathfrak{E}\,, \quad [\mathfrak{G},B] = \mathfrak{G}
\end{aligned}
$$



Commutation relations between generators of B:

$$
\begin{array}{lll}
[T_0^{(1)}, T_1^{(1)}] = T_0^{(1)} , & [T_0^{(1)}, T_2^{(2)}] = 0 , & [T_0^{(1)}, T_3^{(3)}] = 0 , \\
[T_0^{(1)}, J_0] = T_0^{(1)} , & [T_0^{(1)}, J_1^+] = T_1^{(1)} + J_0 , & [T_1^{(1)}, T_2^{(2)}] = 0 , \\
[T_1^{(1)}, T_3^{(3)}] = 0 , & [T_1^{(1)}, J_0] = 0 , & [T_1^{(1)}, J_1^+] = J_1^+ , \\
[T_2^{(2)}, T_3^{(3)}] = 0 , & [T_2^{(2)}, J_0] = 0 , & [T_2^{(2)}, J_1^+] = 0 , \\
[T_3^{(3)}, J_0] = 0 , & [T_3^{(3)}, J_1^+] = 0 , & [J_0, J_1^+] = J_1^+
\end{array}
$$

Correspond to

$$B \cong g\ell_2 \oplus \mathcal{R}^{(2)}$$

The h Hamiltonian (Lie algebraic form)

Lie algebraic form for h:

$$\begin{aligned} h = {} & 4\,T_1^{(1)}\,T_0^{(1)} + 24\,T_2^{(2)}\,T_0^{(1)} + 40\,T_3^{(3)}\,T_0^{(1)} - \frac{48}{5}\,T_2^{(2)}\,T_{11}^{(2)} \\ & + \frac{45}{2}\,T_3^{(22)} + \frac{32}{15}\,T_{12}^{(3)}\,T_2^{(2)} - 48\,T_3^{(3)}\,T_{11}^{(2)} - \frac{64}{3}\,T_3^{(3)}\,T_{12}^{(3)} \\ & + \frac{128}{45}\,T_{222}^{(33)} + (6+60\nu)\,T_0^{(1)} - 4\omega\,T_1^{(1)} - \frac{48}{5}(1+5\nu)\,T_{11}^{(2)} \\ & - 12\omega\,T_2^{(2)} - \frac{64}{15}(2+5\nu)\,T_{12}^{(3)} - 20\omega\,T_3^{(3)} \end{aligned}$$

Discrete H_3 system

The existence of the algebraic form of the Hamiltonian allows us to construct a discrete system with a property of isospectrality.

Let us introduce three pairs of finite difference operators

$$\mathcal{D}_i^{(\delta_i)} = \frac{(e^{\delta_i \partial_i} - 1)}{\delta_i} \ ,$$
$$\mathcal{X}_i^{(\delta_i)} = (\tau_i e^{-\delta_i \partial_i}) \ ,$$

where $i = 1, 2, 3$. They realize a three parametric canonical transformation in $3+3$ phase space

$$[\mathcal{D}_i^{(\delta_i)}, \mathcal{D}_j^{(\delta_j)}] = 0 \ , \ [\mathcal{X}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = 0 \ , \ [\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = \delta_{ij} \ .$$

Discrete H_3 system

The existence of the algebraic form of the Hamiltonian allows us to construct a discrete system with a property of isospectrality.

Let us introduce three pairs of finite difference operators

$$\mathcal{D}_i^{(\delta_i)} = \frac{(e^{\delta_i \partial_i} - 1)}{\delta_i} \ ,$$
$$\mathcal{X}_i^{(\delta_i)} = (\tau_i e^{-\delta_i \partial_i}) \ ,$$

where $i = 1, 2, 3$. They realize a three parametric canonical transformation in $3+3$ phase space

$$[\mathcal{D}_i^{(\delta_i)}, \mathcal{D}_j^{(\delta_j)}] = 0 \ , \ [\mathcal{X}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = 0 \ , \ [\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_j^{(\delta_j)}] = \delta_{ij} \ .$$

Take the linear differential operator $\mathcal{L}(\partial_i, \tau_i)$. Consider the eigenvalue problem

$$\mathcal{L}(\partial_i, \tau_i)\, \varphi(\tau) \;=\; \lambda\, \varphi(\tau)$$

with polynomial solutions

$$\varphi(\tau) = \sum \alpha_{klm} \tau_1^k \tau_2^l \tau_3^m \ .$$

Performing the canonical transformation (discretization) we arrive at

$$\mathcal{L}(\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_i^{(\delta_i)})\ \varphi(\mathcal{X}_i^{(\delta_i)})\ |0\rangle\ =\ \lambda\varphi(\mathcal{X}_i^{(\delta_i)})\ |0\rangle$$

One should introduce the vacuum $|0\rangle$:

$$\mathcal{D}_i^{(\delta_i)}|0\rangle\ =\ 0\ ,\quad i=1,2,3. \qquad \Rightarrow \qquad |0\rangle \equiv 1\ .$$

The corresponding solutions are

$$\tilde{\varphi}(\tau) = \sum \alpha_{klm}\tau_1^{(k)}\tau_2^{(l)}\tau_3^{(m)}\ ,$$

where

$$\tau_i^{(n+1)} = \tau_i(\tau_i - \delta_i)(\tau_i - 2\delta_i)\cdots(\tau_i - n\delta_i) \qquad \textit{(quasi-monomial)}.$$

Performing the canonical discretization for the H_3 Hamiltonian in the algebraic form h_{H_3} we arrive at the isospectral finite-difference operator

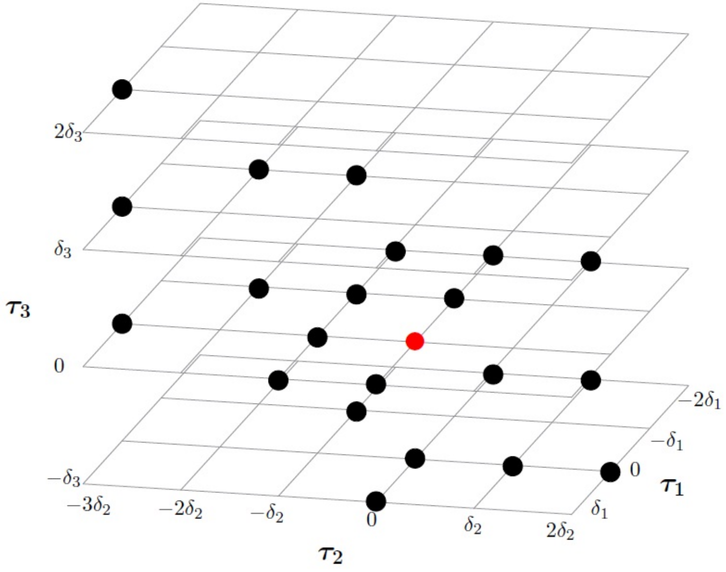
$$\tilde{h}_{H_3} \equiv h_{H_3}(\mathcal{D}_i^{(\delta_i)}, \mathcal{X}_i^{(\delta_i)}) = \sum_{k_1,k_2,k_3} A_{k_1,k_2,k_3}\ e^{k_1\delta_1\partial_1 + k_2\delta_2\partial_2 + k_3\delta_3\partial_3}$$

It is a 22-point finite-difference operator with the following non-vanishing coefficients

$$
\begin{aligned}
A_{0,0,0} &= -\frac{4}{\delta_1}(2+\delta_1\omega)\left[\frac{\tau_1}{\delta_1}+\frac{3\tau_2}{\delta_2}+\frac{5\tau_3}{\delta_3}\right]-\frac{6}{\delta_1}(1+10\nu)\ ,\\
A_{1,0,0} &= \frac{2}{\delta_1}\left[\frac{2\tau_1}{\delta_1}+\frac{12\tau_2}{\delta_2}+\frac{20\tau_3}{\delta_3}+3(1+10\nu)\right]\ ,\\
A_{-1,0,0} &= \frac{4}{\delta_1^2}(1+\delta_1\omega)\tau_1\ ,\\
A_{-2,0,0} &= \frac{48}{5\delta_2}\tau_1(\tau_1-\delta_1)\left[\frac{2\tau_2}{\delta_2}+\frac{5\tau_3}{\delta_3}+1+5\nu\right]\ ,\\
A_{0,-1,0} &= \frac{12}{\delta_1\delta_2}(2+\delta_1\omega)\tau_2\ ,\\
A_{0,0,-1} &= \frac{5}{2}\left[\frac{8}{\delta_1\delta_3}(2+\delta_1\omega)+\frac{9}{\delta_2^2}\right]\tau_3\ ,\\
A_{0,-3,0} &= \frac{128}{45\delta_3^2}\tau_2(\tau_2-\delta_2)(\tau_2-2\delta_2)\ ,
\end{aligned}
$$

$$
\begin{aligned}
A_{1,-1,0} &= -\frac{24\tau_2}{\delta_1\delta_2} \ , \\
A_{1,0,-1} &= -\frac{40\tau_3}{\delta_1\delta_3} \ , \\
A_{-1,-1,0} &= -\frac{32}{15\delta_3}\tau_1\tau_2\left[\frac{\tau_2}{\delta_2} - \frac{20\tau_3}{\delta_3} - 5(1+2\nu)\right] \ , \\
A_{-1,-2,0} &= \frac{32}{15\delta_2\delta_3}\tau_1\tau_2(\tau_2 - \delta_2) \ , \\
A_{-1,-1,1} &= \frac{32}{15\delta_3}\tau_1\tau_2\left[\frac{\tau_2}{\delta_2} - \frac{10\tau_3}{\delta_3} - 5(1+2\nu)\right] \ , \\
A_{-1,-1,-1} &= -\frac{64}{3\delta_3^2}\tau_1\tau_2\tau_3 \ , \\
A_{-1,-2,1} &= -\frac{32}{15\delta_2\delta_3}\tau_1\tau_2(\tau_2 - \delta_2) \ ,
\end{aligned}
$$

$$
\begin{aligned}
A_{-2,1,0} &= -\frac{48}{5\delta_2}\tau_1(\tau_1-\delta_1)\left[\frac{\tau_2}{\delta_2}+\frac{5\tau_3}{\delta_3}+1+5\nu\right] ,\\
A_{-2,-1,0} &= -\frac{48}{5\delta_2^2}\tau_2\tau_1(\tau_1-\delta_1) ,\\
A_{-2,0,-1} &= -\frac{48}{\delta_2\delta_3}\tau_3\tau_1(\tau_1-\delta_1) ,\\
A_{-2,1,-1} &= \frac{48}{\delta_2\delta_3}\tau_3\tau_1(\tau_1-\delta_1) ,\\
A_{0,1,-1} &= -\frac{45\tau_3}{\delta_2^2} ,\\
A_{0,2,-1} &= \frac{45\tau_3}{2\delta_2^2} ,\\
A_{0,-3,1} &= \frac{256}{45\delta_3^2}\tau_2(\tau_2-\delta_2)(\tau_2-2\delta_2) ,\\
A_{0,-3,2} &= \frac{128}{45\delta_3^2}\tau_2(\tau_2-\delta_2)(\tau_2-2\delta_2) .
\end{aligned}
$$


$2\delta_3$
δ_3
τ_3
0
$-\delta_3$
$-3\delta_2$
$-2\delta_2$
$-\delta_2$
0
δ_2
$2\delta_2$
τ_2
$-2\delta_1$
$-\delta_1$
0
τ_1
δ_1

Quasi-exactly-solvable generalization

Among the eigenfunctions of the Hamiltonian h_{H_3} there is an infinite family of eigenfunctions depending on the variable τ_1.

They are solutions of the equation

$$-h_1\ \varphi \equiv -4\tau_1 \frac{\partial^2\varphi}{\partial\tau_1^2} + (4\omega\tau_1 - 6(1+10\nu))\frac{\partial\varphi}{\partial\tau_1}\ =\ \epsilon\varphi\ .$$

Corresponding eigenfunctions are given by Laguerre polynomials:

$$\varphi_{n_1}(\tau_1)\ =\ L_{n_1}^{(1/2+15\nu)}(\omega\tau_1)\ ,\quad \epsilon_{n_1} = 4\omega n_1\ ,\quad n_1 = 0, 1, 2, \ldots$$

The operator h_1 can be rewritten in terms of the generators of the Cartan subalgebra of $sl(2)$ realized by the operators

$$J_k^+ = \tau_1^2 \frac{\partial}{\partial \tau_1} - k\tau_1 \ , \quad J_k^0 = \tau_1 \frac{\partial}{\partial \tau_1} - \frac{k}{2} \ , \quad J^- = \frac{\partial}{\partial \tau_1} \ ,$$

These generators have a common invariant subspace

$$\mathcal{P}_k = \langle \tau_1^p \mid 0 \leq p \leq k \rangle \ , \quad \dim \mathcal{P}_k = (k+1)$$

The operator h_1 takes the $sl(2)$-Lie-algebraic form

$$h_1 = 4J_0^0 J^- - 4\omega J_0^0 + 6(1+10\nu)J^- \ .$$

It preserves the infinite flag of spaces of polynomials

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k \subset \cdots \ ,$$

and any eigenfunction is an element of the flag.

The operator h_1 can be rewritten in terms of the generators of the Cartan subalgebra of $sl(2)$ realized by the operators

$$J_k^+ = \tau_1^2 \frac{\partial}{\partial \tau_1} - k\tau_1 \ , \quad J_k^0 = \tau_1 \frac{\partial}{\partial \tau_1} - \frac{k}{2} \ , \quad J^- = \frac{\partial}{\partial \tau_1} \ ,$$

These generators have a common invariant subspace

$$\mathcal{P}_k = \langle \tau_1^p \mid 0 \leq p \leq k \rangle \ , \quad \dim \mathcal{P}_k = (k+1)$$

The operator h_1 takes the $sl(2)$-Lie-algebraic form

$$h_1 = 4J_0^0 J^- - 4\omega J_0^0 + 6(1+10\nu)J^- \ .$$

It preserves the infinite flag of spaces of polynomials

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k \subset \cdots \ ,$$

and any eigenfunction is an element of the flag.

Construction

- Look for the QES Hamiltonian in the form

$$\mathcal{H}^{(qes)} = \mathcal{H} + V^{(qes)}(\tau_1)$$

- Gauge rotate: $h^{(qes)} = -2(\Psi_0)^{-1}(\mathcal{H}^{(qes)} - E_0)(\Psi_0)$
- $h^{(qes)}$ should possess a τ_1-depending family of eigenfunctions
- One obtains the equation

$$-h_1^{(qes)}\varphi \equiv -4\tau_1\frac{\partial^2\varphi}{\partial\tau_1^2}+(4\omega\tau_1-6(1+10\nu))\frac{\partial\varphi}{\partial\tau_1}+2V^{(qes)}(\tau_1)\varphi = \epsilon\varphi$$

Construction

- Look for the QES Hamiltonian in the form

$$\mathcal{H}^{(qes)} = \mathcal{H} + V^{(qes)}(\tau_1)$$

- Gauge rotate: $h^{(qes)} = -2(\Psi_0)^{-1}(\mathcal{H}^{(qes)} - E_0)(\Psi_0)$
- $h^{(qes)}$ should possess a τ_1-depending family of eigenfunctions
- One obtains the equation

$$-h_1^{(qes)}\varphi \equiv -4\tau_1\frac{\partial^2\varphi}{\partial\tau_1^2}+(4\omega\tau_1-6(1+10\nu))\frac{\partial\varphi}{\partial\tau_1}+2V^{(qes)}(\tau_1)\varphi = \epsilon\varphi$$

Construction

- Look for the QES Hamiltonian in the form

$$\mathcal{H}^{(qes)} = \mathcal{H} + V^{(qes)}(\tau_1)$$

- Gauge rotate: $h^{(qes)} = -2(\Psi_0)^{-1}(\mathcal{H}^{(qes)} - E_0)(\Psi_0)$
- $h^{(qes)}$ should possess a τ_1-depending family of eigenfunctions
- One obtains the equation

$$-h_1^{(qes)}\varphi \equiv -4\tau_1\frac{\partial^2\varphi}{\partial\tau_1^2}+(4\omega\tau_1-6(1+10\nu))\frac{\partial\varphi}{\partial\tau_1}+2V^{(qes)}(\tau_1)\varphi = \epsilon\varphi$$

Construction

- Look for the QES Hamiltonian in the form

$$\mathcal{H}^{(qes)} = \mathcal{H} + V^{(qes)}(\tau_1)$$

- Gauge rotate: $h^{(qes)} = -2(\Psi_0)^{-1}(\mathcal{H}^{(qes)} - E_0)(\Psi_0)$
- $h^{(qes)}$ should possess a τ_1-depending family of eigenfunctions
- One obtains the equation

$$-h_1^{(qes)}\varphi \equiv -4\tau_1\frac{\partial^2\varphi}{\partial\tau_1^2} + (4\omega\tau_1 - 6(1+10\nu))\frac{\partial\varphi}{\partial\tau_1} + 2V^{(qes)}(\tau_1)\varphi = \epsilon\varphi$$

- Let us gauge rotate $h_1^{(qes)}$:

$$h_1^{(sl(2)-qes)} = \tau_1^{-\gamma} \exp\left(\frac{a}{4}\tau_1^2\right) h_1^{(qes)} \tau_1^{\gamma} \exp\left(-\frac{a}{4}\tau_1^2\right)$$

- If $V^{(qes)}$ is chosen of the form

$$V^{(qes)} = \frac{1}{2}\, a^2\tau_1^3 + a\omega\tau_1^2 - a\left(2k + 2\gamma + 15\nu + \frac{5}{2}\right)\tau_1 + \frac{\gamma(2\gamma + 30\nu + 1)}{\tau_1}$$

then $h_1^{(sl(2)-qes)}$ is in $sl(2)$-Lie-algebraic form:

$$h_1^{(sl(2)-qes)} = 4J_k^0 J^- - 4aJ_k^+ - 4\omega J_k^0 + 2(k + 4\gamma + 3(1 + 10\nu))J^- .$$

- Let us gauge rotate $h_1^{(qes)}$:

$$h_1^{(sl(2)-qes)} = \tau_1^{-\gamma} \exp\left(\frac{a}{4}\tau_1^2\right) h_1^{(qes)} \tau_1^{\gamma} \exp\left(-\frac{a}{4}\tau_1^2\right)$$

- If $V^{(qes)}$ is chosen of the form

$$V^{(qes)} = \frac{1}{2}\, a^2 \tau_1^3 + a\omega\tau_1^2 - a\left(2k + 2\gamma + 15\nu + \frac{5}{2}\right)\tau_1 + \frac{\gamma(2\gamma + 30\nu + 1)}{\tau_1}$$

then $h_1^{(sl(2)-qes)}$ is in $sl(2)$-Lie-algebraic form:

$$h_1^{(sl(2)-qes)} \;=\; 4J_k^0 J^- - 4aJ_k^+ - 4\omega J_k^0 + 2(k + 4\gamma + 3(1 + 10\nu))J^- \;.$$

The operator $h_1^{(sl(2)-qes)}$ has $\mathcal{P}_k$ as an invariant subspace, but it does not preserve a flag of spaces.

It has $(k+1)$ polynomial eigenfunctions of the form of polynomials of the degree k,

$$P_j^{(k)}(\tau_1) = \sum_{i=0}^{k} \gamma_i^{(j)} \tau_1^i \ , \quad j = 0, 1, 2, \dots \ ,$$

while other eigenfunctions are not polynomials.

$sl(2)$-quasi-exactly-solvable Hamiltonian associated with the root space H_3:

$$\begin{aligned}\mathcal{H}^{(qes)} &= \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2}+\omega^2 x_k^2+\frac{g}{x_k^2}\right] \\ &+ \sum_{\{i,j,k\}}\sum_{\mu_{1,2}=0,1}\frac{2g}{[x_i+(-1)^{\mu_1}\varphi_+ x_j+(-1)^{\mu_2}\varphi_- x_k]^2} \\ &+ \frac{1}{2}\,a^2(\mathbf{x}^2)^3+a\omega(\mathbf{x}^2)^2-a\left(2k+2\gamma+15\nu+\frac{5}{2}\right)\mathbf{x}^2 \\ &+ \frac{\gamma(2\gamma+30\nu+1)}{\mathbf{x}^2}\ ,\end{aligned}$$

where $\{i,j,k\}=\{1,2,3\}$ and all even permutations, and $\mathbf{x}^2=\sum_{i=1}^{3}x_i^2$.

We know $(k+1)$ eigenstates explicitly. Their eigenfunctions are of the form

$$\Psi_k(x) \;=\; \Delta_1^{\nu}\Delta_2^{\nu}\,(\mathbf{x}^2)^{\gamma}\cdot P_k(\mathbf{x}^2)\,\mathrm{e}^{-\frac{\omega}{2}\mathbf{x}^2-\frac{a}{4}(\mathbf{x}^2)^2}$$

where P_k is a polynomial of degree k and $g=\nu(\nu-1)>-\frac{1}{4}$.

Conclusion

- An algebraic form for the H_3 rational model exists. It acts on the spaces of polynomials $\mathcal{P}_n^{(1,2,3)}$, $n = 0, 1, 2, \ldots$ Eigenfunctions are elements of these spaces
- An integral of motion exists. It has an algebraic form in τ variables
- The hidden algebra of the H_3 model is the $h^{(3)}$ algebra, which has infinite dimension but is finitely generated
- It is possible to construct an isospectral discrete model and a quasi-exactly-solvable generalization
- Other integral(s) of motion has not been found yet
- Why different minimal flags are preserved by h and f?

$$\tau_1 = x_1^2 + x_2^2 + x_3^2\,,$$

$$\begin{aligned}\tau_2 = & -\frac{3}{10}\,(x_1^6 + x_2^6 + x_3^6) + \frac{3}{10}(2-5\varphi_+)\,(x_1^2 x_2^4 + x_2^2 x_3^4 + x_3^2 x_1^4) \\ & + \frac{3}{10}(2-5\varphi_-)\,(x_1^2 x_3^4 + x_2^2 x_1^4 + x_3^2 x_2^4) - \frac{39}{5}\,(x_1^2 x_2^2 x_3^2)\,,\end{aligned}$$

$$
\begin{aligned}
\tau_3 = {} & \frac{2}{125}\left(x_1^{10}+x_2^{10}+x_3^{10}\right)+\frac{2}{25}(1+5\varphi_-)\left(x_1^8x_2^2+x_2^8x_3^2+x_3^8x_1^2\right)\\
&+\frac{2}{25}(1+5\varphi_+)\left(x_1^8x_3^2+x_2^8x_1^2+x_3^8x_2^2\right)\\
&+\frac{4}{25}(1-5\varphi_-)\left(x_1^6x_2^4+x_2^6x_3^4+x_3^6x_1^4\right)\\
&+\frac{4}{25}(1-5\varphi_+)\left(x_1^6x_3^4+x_2^6x_1^4+x_3^6x_2^4\right)\\
&-\frac{112}{25}\left(x_1^6x_2^2x_3^2+x_2^6x_3^2x_1^2+x_3^6x_1^2x_2^2\right)\\
&+\frac{212}{25}\left(x_1^2x_2^4x_3^4+x_2^2x_3^4x_1^4+x_3^2x_1^4x_2^4\right).
\end{aligned}
$$

Back to presentation

$$
\begin{aligned}
\mathcal{F} = & -\frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+\frac{2\nu(\nu-1)}{(s_\theta c_\phi+\varphi_+ s_\theta s_\phi+\varphi_- c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\theta c_\phi-\varphi_+ s_\theta s_\phi+\varphi_- c_\theta)^2}+\frac{2\nu(\nu-1)}{(s_\theta c_\phi+\varphi_+ s_\theta s_\phi-\varphi_- c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\theta c_\phi-\varphi_+ s_\theta s_\phi-\varphi_- c_\theta)^2}+\frac{2\nu(\nu-1)}{(s_\theta s_\phi+\varphi_+ c_\theta+\varphi_- s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\theta s_\phi-\varphi_+ c_\theta+\varphi_- s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(s_\theta s_\phi+\varphi_+ c_\theta-\varphi_- s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\theta s_\phi-\varphi_+ c_\theta-\varphi_- s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\theta+\varphi_+ s_\theta c_\phi+\varphi_- s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\theta-\varphi_+ s_\theta c_\phi+\varphi_- s_\theta s_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\theta+\varphi_+ s_\theta c_\phi-\varphi_- s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\theta-\varphi_+ s_\theta c_\phi-\varphi_- s_\theta s_\phi)^2}+\frac{\nu(\nu-1)}{2s_\theta^2 c_\phi^2}+\frac{\nu(\nu-1)}{2s_\theta^2 s_\phi^2}+\frac{\nu(\nu-1)}{2c_\phi^2}\ .
\end{aligned}
$$

▸ Back to presentation